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การวาดลายไทยบนพื้นผิววัตถุ 3 มิติ

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บทคัดย่อ บทความนี้เสนอการวาดลายไทยบนพื้นผิววัตถุ 3 มิติ ลายไทยซึ่งสวยงามและสลับซับซ้อนได้ถูกแยกเป็นรูปลายไทยพื้นฐาน (primitive) 11 ชนิด คือ กระจัง พุ่มข้าวบิณฑ์ กนก กนกเปลว กาบ ตัวเหงา ก้นหอย วงกลม วงรี สี่เหลี่ยมผืนผ้า รูปหลายเหลี่ยม สำหรับรูปลายไทยพื้นฐานที่เป็นลักษณะโค้ง เช่น กระจัง กนก จะถูกสร้างด้วยเส้นโค้ง Bezier ทำให้สามารถแปลงรูปได้ ทั้งการเปลี่ยนขนาดรูป การหมุนรูป การบิดรูป และการสะท้อนรูป ด้วยการใช้รูปลายไทยพื้นฐานเหล่านี้ ผู้ใช้สามารถผูกเป็นลายไทยซึ่งสลับซับซ้อนและสวยงามได้ตามต้องการ หลังจากนั้นรูปลายไทยที่วาดบนพื้นผิว 2 มิติ จึงถูกทาบ (texture mapping) ลง บนพื้นผิวของวัตถุ 3 มิติ ด้วยการแบ่งพื้นผิววัตถุ 3 มิติ ออกเป็นพื้นผิวเล็ก ๆ จึงทำให้ทาบลายไทยบนผิวโค้งของวัตถุ 3 มิติได้ ในการทดลองได้แสดงรูปลายไทยบนพื้นผิววัตถุ 3 มิติ ในมุมมองต่าง ๆ และเนื่องจากมีการจัดการการบังกันของพื้นผิว (hidden surface) จึงทำให้รูปที่ได้ดูสวยงามและสมจริง

1. บทนำ

ลายไทยเป็นศิลปของไทยมาตั้งแต่โบราณ การหัดเขียนลายไทยเริ่มจากการเขียนลายง่าย ๆ แล้วต่อลายที่ยากขึ้นไปตามลำดับ แล้วจึงคิดประดิษฐ์ผูกลายขึ้นตามจินตนาการของผู้เขียนเอง ในการวิจัยนี้ได้ศึกษาวิเคราะห์ลายไทยจากลวดลายที่ปรากฏอยู่ในที่ต่าง ๆ ตลอดจนคำปรึกษาจากอาจารย์มหาวิทยาลัยศิลปากร จึงกล่าวได้ว่าลายไทยโดยทั่วไปประกอบด้วยองค์ประกอบพื้นฐาน (primitive) เพียงไม่กี่ชนิด แต่การผูกลายให้ได้ความงดงามแตกต่างกันนั้นขึ้นอยู่กับฝีมือ จินตนาการ และความชำนาญของผู้เขียนลายไทยที่จะนำองค์ประกอบพื้นฐานเหล่านี้ไปผูกกันเป็นลวดลายที่งดงาม ซึ่งองค์ประกอบพื้นฐานของลายไทยกล่าวได้ว่ามี 11 ชนิด คือ กระจัง พุ่มข้าวบิณฑ์ กนก กนกเปลว กาบ ตัวเหงา ก้นหอย วงกลม วงรี สี่เหลี่ยมผืนผ้า และรูปหลายเหลี่ยม องค์ประกอบพื้นฐานเหล่านี้ส่วนใหญ่ประกอบด้วยเส้นโค้ง และด้วยหลักการของคอมพิวเตอร์กราฟฟิกนั้น สามารถสร้างเส้นโค้งซึ่งสลับซับซ้อนได้ โดยอาศัย spline curve หรือ Bezier curve บทความนี้จึงเสนอการสร้างองค์ประกอบพื้นฐานของลายไทยด้วยหลักการของคอมพิวเตอร์กราฟฟิกซึ่งทำให้สามารถทำการแปลงรูปเหล่านี้ได้ไม่ว่าเป็นการเปลี่ยนขนาดรูป การหมุนรูป การบิดรูป และการสะท้อนรูป ด้วยการใช้องค์ประกอบพื้นฐานเหล่านี้รวมทั้งเทคนิคการแปลงรูป ผู้ใช้สามารถผูกเป็นลายไทยที่งดงาม และสลับซับซ้อนได้ตามจินตนาการ

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and the assignment problem. The particular procedures which were used in the BASIC program for the ADP are not necessarily the most efficient.

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5. Conclusions, Remarks and Extensions

This research has proposed two heuristic methods for solving the MTSP with balancing workloads. Balancing workloads can be described by two criteria: (1) minimizing the maximal tour and (2) minimizing the range among tours. The secondary objective of minimizing the total distance is also considered. Discussion of results in section 4 showed that sometimes it is difficult to optimize these objectives simultaneously. For instance, the ADP usually provides a better solution of minimizing the total distance and the maximal tour while the AEH provides a solution with smaller range among tours. The choice of solution technique will depend on the priority or weighting factor of each objective. This can be defined from observations and investigation of the real world problem or the opportunity cost compared with the lower bound of each objective. Computational time is also a constraint, especially in a dynamic system where the problem may change (as new jobs are added) before the solution can be implemented. For this reason, the ADP, in some cases, may not be appropriate when the problem becomes large. Experiments with real data also help to justify the selection between methods.

All experimentation in this research have been done by microcomputers which is considerably a cheaper investment comparing with using a mini or main frame computer. We have been able to solve an intermediate scale model of 50 jobs and 5 vehicles in a reasonable time in some cases.

Another point which should be mentioned is the fact that the cost matrix in FMHS is asymmetric. This is the reason why we have been able to applied the modified Eastman's algorithm in the ADP successfully. For a symmetrical cost matrix, the method described in Held and Karp (1970, 1971) is probably a better approach in solving the MTSP for the ADP.

Extension of this research could be conducted in the following ways. First, both the ADP and the AEH can be modified to handle capacity and demand constraints of the VRP. An area of application would be the multiple unit load FMHS which allows each vehicle to have a multi-unit load capacity. The AEH could be modified easily by adding a routine to detect the tour which violates the capacity constraints as an infeasible tour. A modification of the ADP could be made to change the MTSP to the VRP and find an appropriate method for solving the VRP. The method in Stewart and Golden (1984) would be one possible choice.

An interactive integer programming approach could be tested to provide a better solution and provide a better lower bound than those used in section 4. The integer linear program in section 1.1 could be reformulated as a goal program and solved by modifying methods described in Groschel (1980) and Padberg and Hong (1980). A good commercial LP code should be available.

This research could be focused on multi-criteria decision making. A more rigorous approach to multi criteria optimization could be undertaken.

The effect of the initial tours (step 1) to the final solution of the AEH should be investigated and to determine which method generates the initial tour that leads to the best solution in the fewest iterations.

The time window constraints could be added into the previous FMHS model and studied. These constraints represent time intervals which each department open to pick up or deliver loads. Both exact procedures and heuristics could be developed.

A comparison to two methods described in Husban (1985) and other methods (if available) could be done and finally, the computer programs developed for the ADP and the AEH are not particular efficient code; certainly there are improvements which can be made in this area to decrease execution time. For instance, there are many methods to solve the MTSP produces a better OPRM and OPRT values, while the AEH produces a better OPRD value, which supports the results from Table 1 to Table 4.

Table 3 Comparisons of the OPRD values

Size Levels		Methods	
		ADP	AEH
20-3	Mean	1.16876	1.03329
	S.D.	0.12955	0.03478
	Min.	1.00719	1.00000
	Max.	1.54362	1.22951
35-4	Mean	1.25046	1.02433
	S.D.	0.16694	0.01910
	Min.	1.01695	1.00000
	Max.	1.87135	1.09551
50-5	Mean	1.23842	1.02354
	S.D.	0.13354	0.01921
	Min.	1.05263	1.06322
	Max.	1.83938	1.07456

Table 4 Comparisons of the running time (sec.)

Size Levels		Methods	
		ADP	AEH
20-3	Mean	120.34000	70.17000
	S.D.	53.09364	13.81996
	Min.	35.00000	41.00000
	Max.	282.00000	114.00000
35-4	Mean	462.11000	318.57000
	S.D.	193.79400	58.52738
	Min.	100.00000	209.00000
	Max.	972.00000	467.00000
50-5	Mean	1190.60000	922.08000
	S.D.	577.14460	161.66910
	Min.	160.00000	602.00000
	Max.	3056.00000	1513.00000

Table 5 Comparisons Replication by Replication Between the ADP and the AEH

Size	Method	% of favorable replication by replication		
		OPRM	OPRT	OPRD
20-3	ADP	58	100	9
	AEH	29	0	91
35-4	ADP	47	100	2
	AEH	43	0	98
50-5	ADP	58	100	0
	AEH	34	0	100

not be true under some cost structures, because sometimes decreasing the maximal tour may affect the total distance. However, from the previous results we can conclude that the generated cost matrix has several solutions with minimum total distance.

Table 1 Comparisons of The OPRM Value

Size Levels		Methods	
		ADP	AEH
20-3	Mean	1.16876	1.20604
	S.D.	0.12955	0.09653
	Min.	1.00719	1.02273
	Max.	1.54362	1.49485
35-4	Mean	1.25046	1.22834
	S.D.	0.16694	0.09980
	Min.	1.01695	1.05114
	Max.	1.87135	1.49007
50-5	Mean	1.23842	1.27608
	S.D.	0.13354	0.10220
	Min.	1.05263	1.06322
	Max.	1.83938	1.56463

Table 2 Comparisons of The OPRT Value

Size Levels		Methods	
		ADP	AEH
20-3	Mean	1.00087	1.16876
	S.D.	0.00751	0.08476
	Min.	1.00000	1.02532
	Max.	1.07407	1.43750
35-4	Mean	1.00000	1.19993
	S.D.	0.00000	0.09019
	Min.	1.00000	1.04286
	Max.	1.00000	1.50000
50-5	Mean	1.00000	1.24815
	S.D.	0.00000	0.09429
	Min.	1.00000	1.06358
	Max.	1.00000	1.53425

In Table 3, the OPRD values of both methods are compared. The results show that the AEH produces a solution with a smaller (on average) deviation. This results support the fact the AEH reduces the maximal tour by increasing the length of the minimal tour or some remaining tours.

In Table 4, a comparison of running time is studied. It appears that the AEH produce a good solution more quickly. This is especially true when the problem is large.

Another comparison between both methods is shown in table 5 which compares the result among objectives replication by replication and shows the relative performance measured in each size level. Charnsethikul (1986) used the sign-test (see Conover (1971)) in Table 5 to test whether the ADP is preferred to the AEH and the conclusion is that the ADP

4. Computational Experiences

To evaluate both algorithms in the previous chapter, 100 replications with three different size levels of the FMHS routing problem from the experimentation of Blair, Charnsethikul and Vasquez (1985) were used. This data set was generated by using the random number generator of the TI Basic language. The data input consists of the distance matrix among departments in the system and a list of jobs which contains the origin and destination of each job. The cost matrix in each replication is generated by the method described in section 1.2. It is appeared that all generated cost matrices are asymmetric. The size levels of the problems are 20 jobs with 3 vehicles (level I), 35 jobs with 4 vehicles (level II), and 50 jobs with 5 vehicles (level III). Both algorithms were coded in Basic and were complied by the Microsoft Basic compiler version 2.1 for the TI microcomputer. Evaluations of the solution in each replication contain four performance measurements:

1) The optimization performance ratio of the maximal tour (OPRM) is defined as the ratio of the maximal tour at termination to a lower bound which is determined by averaging the tour values from the solution of the MTSP in step 1 of the ADP. In other word, this lower bound is calculated from dividing the total distance of the MTSP by the number of vehicles. The actual lower bound or the exact maximal tour is possibly greater than this approximated lower bound since the implied perfectly balanced allocation may not be possible. The result is that the cited OPRM value can be considered as the worst case performance".

2) The optimization performance ratio of the total distance (OPRT) is defined as the ratio of the total distance at termination to the total distance of the MTSP solution in step 1 of the ADP.

3) The optimization performance ratio of deviation among tours (OPRD) is defined as the ratio of the maximal tour at termination to the averaged value of total distance in each tour at termination.

4) Running time (T) is estimated in the unit of seconds.

Comparisons of performance measurements between the ADP and the AEH is summarized from Table 1 to Table 4. In each table, the mean value, the standard deviation, the maximum and the minimum in each performance measurement of both methods in all sizes are shown.

In Table 1, the OPRM values are compared. It is found that the ADP has a better average OPRM value but its average standard deviation is larger. Considering the worst case performance in the OPRM value (its maximum value) of the ADP to those of the AEH, we found that the average worst case of the AEH is 51 percent comparing to 75 percent of the ADP. Charnsethikul (1986) used a histogram to represent the distribution of the OPRM value in each size level. He discovered an ill condition of the ADP when more than one maximal tour occurs or the total distance of some other tour are close to that of the maximal tour. He suggested some modification of the ADP to correct this condition but the approach is costly in total computational time.

In Table 2, a comparison of the OPRT values shows that the ADP usually produce a better solution with respect to minimizing the total distance. This is obviously true because we use the modified Eastman's algorithm which minimizes the total distance in each subproblem of the ADP. In this experiment, the OPRT value in almost every replication solved by the ADP is equal to 1. This means that the ADP often produces the solution with minimum total distance and tends to minimize the maximal tour at the same time. Nevertheless, this fact may

This solution technique would appear to require a lot of computational time for solving a sequence of the MTSP. In fact, the algorithm uses the final cost matrix of the previous iteration as the starting point to continue searching, instead of starting at the original cost matrix again. It does not take a long time to produce another tour when it is compared to the time spent in step 1.

3.2 Arc Exchanging Heuristic (AEH)

One of the better heuristic for the TSP is the 3-OPT procedure which was introduced originally by Lin and Kernighan (1973). To minimize the longest tour duration, Blair and Vasquez (1984) suggested a technique which transfers a node from the maximal tour to the remaining tour which they refer to as "MIN-MAX KICK OUT ". AEH is a combination of the above two methods with additional objective of minimizing the longest deviation among tours.

The basic procedure of the 3-OPT algorithm is to find a better solution after exchanging three arcs from the old tour. There are many rules to select these three arcs. In this case, the selection rule is based on a node exchanging procedure start with a feasible tour. Let arcs (a,b), (c,d), (e, f) be arcs in a feasible tour where arc (i, j) means that point j immediately follows point i. Consider the substitution the three new arcs of (a,d), (c, f), (e,b) to the previous tour. The new total cost of the objective function is calculated. The method of enumeration is used to select these three arcs from among n arcs. where n equals the total arcs in the old tour. There are $n(n-1)(n-2)/6$ possible sets of three arcs to be substituted into the old tour. Whenever any improvement is found. the procedure starts again and stop when no improvement occurs. Since there are several way to exchange three arcs, there may be other modification which could produce a better solution than the method presented.

To satisfy balancing workloads, our objective function is to minimize the longest tour and the longest deviation among tours instead of the total distance. An additional heuristic which is added to the 3-OPT procedure is to select the maximal tour first and then the minimal tour to perform the 3-OPT procedure for all possible combinations. If there is no improvement, the process will continue searching by applying the 3-OPT procedure to all tours. It is obvious to see that this heuristic tends to minimize the range among tours and minimize the maximal tour regardless of the total distance.

To simplify the above idea, the new heuristic method can be described as follows.

Step 1. Find a feasible solution for the MTSP.

Step 2. Identify the minimal tour and the maximal tour.

Step 3. Apply the 3-OPT procedure to the tours in step 2. If no improvement in the maximal tour has been found, go to step 4, otherwise go to step 2.

Step 4. Apply the 3-OPT procedure to all tours.

Step 5. If no improvement in the maximal tour has been found in step 4, stop and print the best solution found, otherwise, go to step 2.

This algorithm consists of two phases. In the first phase, the minimal tour and the maximal tour are identified in step 2 after an overall tour was found. The 3-OPT procedure used in step 3 tends to decrease the maximal tour and increase the minimal tour. The second phase uses the 3-OPT procedure in step 4 to decrease the maximal tour and increase some of the remaining tours when no improvement is found in step 3.

In recent years, a good deal of work has been done in the development of heuristic programs for solving large combinatorial problems. According to the definition from Weist (1977) a heuristic is a method of reducing the search in a problem solving situation as an aid to the discovery of a solution. The phrase "rule of thumb" often is used synonymously with "heuristic". In other words, a collection of rules of thumb for solving a particular problem is called a heuristic program. If sufficiently complex, such a program may require a computer for its solution. Heuristic programs for the MTSP with balancing workloads can be separated into two forms:

1) Arc Deleting Procedure(ADP): this attempts to reduce the longest tour and maintain minimization of the total distance.
2) Arc Exchanging Heuristic(AEH): this attempts to reduce the longest tour and the longest deviation among tours.

3.1 An Arc Deleting Procedure (ADP)

The ADP is an appropriate modification to Eastman's algorithm. For the MTSP, Eastman's algorithm has some advantages over Little's algorithm. For instance, there is no difference in the level of branching or fathoming between solving the MTSP and the TSP using Little's algorithm, because the algorithm treats the MTSP the same as the TSP (an illustration is in Dean & White (1975)). Eastman's algorithm has a difference rule to fathom an active node; it considers whether the tour is feasible or not for the MTSP. For example, consider a problem with five nodes and two vehicles. Suppose that the solution from the assignment problem is 6-1-2-6, and 7-3-5-4-7, where 6,7 represent the starting point of each vehicle. This tour is feasible for the MTSP, but it is not feasible for the TSP. If we solve the MTSP by Eastman's algorithm and use its rule for solving the TSP, we have to continue branching and searching for the solution of the TSP. In fact, we already have the optimum tour in the first step. This was illustrated by Svestka and Huckfeldt (1973) when they modified this rule to Eastman's algorithm. The results showed that solving the MTSP usually required fewer steps than solving the TSP.

To satisfy balancing workloads, a heuristic technique is applied to the problem of minimum total distance solved by the method as described in the previous paragraph. This heuristic is called the "arc deleting procedure". The main goal is to reduce the maximal tour. The procedure deletes each link (i, j) where (i, j) is a sequence of node in the maximal tour, by assigning C_{ij} (cost of traveling from node i to node j) equal to infinity, and the MTSP corresponding to each deletion of link (i, j) is solved. Suppose there are k arcs contained in the maximal tour, thus producing the new k solutions. We select the best improvement to continue searching in the same way until no improvement is found.

From the previous procedure, the algorithm can be described as follows.

Step 1. Solve the MTSP by Eastman's algorithm with the modified rule of recognizing subtours from the TSP as feasible tours of the MTSP.

Step 2. From the solution in step 1 or 3, select the produced maximal tour and start to delete each link (i, j) in that tour and resolve the MTSP corresponding to each deletion of (i, j).

Step 3. If there is no improvement in the maximal tour from step 2, stop and print the best solution, otherwise, select the best improvement as the solution and go to step 2.

Smith (1980) compared results produced by Eastman's algorithm to those of Held and Karp and showed that Eastman's algorithm is better when the cost matrix is asymmetrical.

Balas and Christofides (1981) form Lagrangian relaxation of the TSP by moving appropriate subtour elimination constraints to the objective function and develop a branch and bound scheme by solving a sequence of the assignment problem. It is appeared that this method is the best known for asymmetric problems.

Lin and Kernighan (1973) proposed an effective heuristic procedure for the TSP. The general concept is to transfer arcs which are not included in the previous tour into a new tour by exchanging nodes. They presented several algorithms to show methods which can be used to generate a set of tours from an available tour. A method which is widely used is the 3-OPT procedure. The process is to choose three arcs out of the old tour and find three new arcs to replace them. Several new tours are generated (depend on how many possible three arc sets from the old tour can be chosen). An objective function must be evaluated and the process stops when all new tours show no improvement in the objective value. Otherwise, a tour with improvement is chosen to start the process again. They also presented an additional algorithm to decide at each iteration how many branches to exchange instead of using three branches.

Stewart and Golden (1984) modified the idea of Lin and Kernighan with Lagrangian relaxation to the general VRP by moving the capacity constraints to the objective function. The remaining constraints of the problem constitute the MTSP. The capacity constraints are multiplied by a set of Lagrange multiplier. An initial value for the Lagrange multipliers must be given and the 3-OPT procedure is applied to solve the MTSP with the new objective. They presented an algorithm to adjust the Lagrange multiplier in order to obtain an improved solution.

Dean and White (1975) studied balancing workloads in machine scheduling with the approach of a modified Little's algorithm. The procedure is to continue searching until the best solution with the best balance is found (measured as minimum range). Computational results of some small problems were reported.

Blair and Vasquez (1984) proposed a heuristic which is based on a node exchanging procedure which transfer a node in the maximal tour to the remaining tours in order to minimize the longest tour and the total distance. An application of this method was made to solve the VRP in a Flexible Material Handling System (FMHS). They assumed that all vehicles carried a single unit load at the time they passed through a sequence of jobs, so the VRP becomes the MTSP. Later, Blair, Charnsethikul and Vasquez (1985) developed an alternative stopping rule and a different rule for node selection in the maximal tour which is transferred to the remaining tour. The algorithm was tested at three levels of 15 nodes and 3 vehicles, 35 nodes and 4 vehicles, and 50 nodes and 5 vehicles. One hundred replications were generated at each level.

Husban (1985) formulated the problem of minimizing the maximal tour of the MTSP as an integer linear program and solved it using a branch and bound technique similar to that of Dean and White (1975) for small problems. He also formulated the VRP in the FMHS as a transportation problem by an arc covering approach. Two new heuristic methods were developed. The first heuristic is based on a node covering approach, while the latter heuristic is based on an arc covering method. He reported some numerical experience using the same lower bound developed in Blair and Vasquez (1984) to measure the performance of both heuristics.

3. Solution Techniques

distance traveled from department 2 (the starting location of vehicle 1) to department 2 (the starting location of job 2), and ended at department 3 which is the destination of job 2 (C_{62} = $t(2,2) + t(2,3)$). Similarly, other $c(i, j)$ can be computed. The additional number ("6" and "7") used in this formulation is similar to the one described in Gorenstein (1970) which is used to represent the MTSP. Usually, they are called "Dummy Salesman (Vehicles)". This formulation converts the MTSP to the TSP which several algorithms are available. Modifications of this formulation can also be found in Dean and White (1975) and Svestka and Huckfeldt (1973).

A brief review of existing the TSP and the MTSP is given in section 2. Solution techniques and computational experiences are discussed in section 3 and 4 respectively. Some conclusions and comments about extensions to this research are presented in section 5.

2. Review of The TSP and The MTSP

The TSP has received much attention since 1950. A good survey of theoretical results, solution techniques and applications was presented in Bellmore and Nemhauser (1973), Lawler et al. (1985) and Parker and Rardin (1983).

Grotschel (1980) demonstrated how knowledge of the facets of the polytope associated with the symmetric TSP can be utilized to solve the large scale TSP. In particular, he reported how the shortest round trip through 120 German cities was found by using a commercial linear programming code and adding facetial cutting planes in an interactive way.

Little's algorithm (1963) is a branch and bound method. The set of all tours (feasible solutions) is broken up into increasingly smaller subsets by a procedure called branching. For each subset, a lower bound on the length of the tours therein is calculated. Eventually, a subset is found that contains a single tour whose length is less than or equal to some lower bound for every tour.

Eastman's algorithm (1958) is also a branch and bound method. The branching process is considered from the subtour elimination constraints. Bellmore and Malone (1971) reported some computational experiences and statistical analysis in both symmetrical and asymmetrical cases.

Gorenstein (1970) gave a formulation of the MTSP as a TSP by adding row and column for each additional tour (salesman) and set the cost coefficient to infinity for linking pairs of these added rows and columns. The cost coefficient of linking pairs between these added rows and columns to the rows and columns in the original cost matrix must be given. He used the MTSP to schedule a printing press for a periodical with several editions in order to minimize the sequence dependent setup costs. He proved that the solution to the MTSP can be obtained from the solution to a TSP with M home visits. This permits the use of any method in the TSP to solve the MTSP by allowing a tour with M home visits to be a feasible tour.

Svestka and Huckfeldt (1973) gave another formulation of the MTSP as the TSP. They proposed the idea of augmenting the cost matrix which was almost the same as that shown in Gorenstein (1980). The algorithm of Bellmore and Malone is used to solve the MTSP with the additional idea of generating an initial tour to be used as an upper bound of the solution. This idea helps to reduce computational time in some cases. They also found that the inclusion of additional salesman can reduce the total computation to a fraction of the time of the one salesman case. For large problems, this method still increases exponentially as the number of cities grows, but the rate of increase is lower than that of Little's algorithm.

Held and Karp (1970,1971) proposed another branch and bound method to solve the symmetric TSP by using the solution from the minimum spanning tree associated with the "From-To" matrix to provide the lower bound and combining concepts of subgradient optimization and Lagrangian relaxation to solve the TSP in a dual form.

formulated and solved by an integer programming code, but the running time is costly and there are some problems of roundoff error and error propagation when the problem is large. This approach is not appropriate in the case of this application.

1.2 Material Handling Systems (An MDMTSP Formulation)

An area of application in this research is the problem of determining the optimal path of automated guided vehicles to support a flexible material handling system (FMHS). From Blair and Vasquez (1984), the problem can be described as follows.

Given a set of jobs J, represented by an origin from which the job must originate, a destination at which the job must terminate, and a matrix of distances among departments in the system, the objective is to find an allocation of jobs to tours which both minimizes and balances the distance which each vehicle must travel. It is assumed that each vehicle carries a single unit load at a time as they pass through a sequence of jobs.

To provide a more mathematical description of the travel cost C_{ij} in section 1.1, it is necessary to define additional notation.

Let $J = \{(o(j), d(j))\}$ denote the set of ordered pairs (jobs) with the element j defined by the job origin $o(j)$ and the jobs destination $d(j)$, $j = 1,\dots, N$. Let $T = [t(i, j)]$ denote the matrix of distances between department pairs.

The "cost" for job j to follow job i in some tour is:

$$C_{ij} = t(o(j), d(j)) + t(d(i), o(j)) \qquad i, j = 1,\dots, N \text{ and } i \neq j \qquad (10)$$

$$= t(o(i\text{-}N), o(j)) + t(o(j), d(j)) \qquad i = N+1,\dots, N+M;\ j = 1,\dots, N \qquad (11)$$

$$= 0 \qquad i = 1,\dots, N;\ j = N+1,\dots, N+M \qquad (12)$$

$$= \infty \qquad i, j = N+1,\dots, N+M \qquad (13)$$

$$= \infty \qquad i = j;\ i, j = 1,\dots, N+M \qquad (14)$$

where

(10) represents the distance from the destination of job i to the destination of job j;

(11) represents the distance from the initial depot of vehicle i-N to the destination of job j;

(12) represents the indifference to the ending point which allows all vehicles to stop at the last destination;

(13) represents the fact that all vehicles are used to perform at least one job; and

(14) represents the fact of the TSP that job i can not be performed twice in the sequence of operations.

An example of formulating the cost matrix of the FMHS is as follows. A set of job consists of 5 jobs of 1(2,1), 2(2,3), 3(1,5), 4(4,2) and 5(4,5) where $k(i, j)$ represents the starting department (i) and the destination department (j) of each job k and the T matrix is given. Suppose there are two vehicles located at departments 2 and 3. Let numbers "6" and "7" represent the starting points of both vehicles which are located at departments 2 and 3 respectively. Suppose we want to compute $c(4,5)$, which means the distance traveled from job 4 to job 5. From the given data, job 4 is ended at department 2, and job 5 is started at department 4 and ended at department 5, so the total distance traveled is the distance traveled from department 2 to department 4 and from department 4 to department 5, which can be written as $C_{45} = t(2,4)+t(4,5)$. Another example is to compute C_{62} which means the total distance traveled from the starting location of vehicle 1 to job 2. The total distance is the efficient heuristic which solves the MDMTSP and produces solutions which are close to the optimal solution of each individual objective.

1.1 An Integer Programming Formulation

The basic mathematical formulation for the MDMTSP with all balancing criteria is an integer program with multiple objectives.

$$Z_1 = \text{Min } T \qquad (1)$$

$$Z_2 = \text{Min} \sum_{i,j,k} C_{ij} X_{ijk}, \qquad i,j = 1,\ldots, N+M,\ k = 1,\ldots, M \qquad (2)$$

$$Z_3 = \text{Min } D \qquad (3)$$

Subject to

$$T \geq \sum_{i,j} C_{ij} X_{ijk}, \qquad k = 1,\ldots, M \qquad (4)$$

$$D \geq \text{ABS}[T(L1) - T(L2)], \qquad L1 \neq L2 \qquad (5)$$

where

$$T(L) = \sum_{i,j} C_{ij} X_{ijl}, \qquad l = 1,\ldots, M$$

$$\sum_{i,k} X_{ijk} = 1 \qquad j = 1,\ldots, N+M \qquad (6)$$

$$\sum_{j,k} X_{ijk} = 1 \qquad i = 1,\ldots, N+M \qquad (7)$$

$$\sum_{i} X_{ihk} - \sum_{j} X_{hjk} = 0, \qquad h = 1,\ldots, N+M,\ k = 1,\ldots, M \qquad (8)$$

$$U_i - U_j + (M+N)\sum_{k} X_{ijk} \leq M + N - 1, \qquad i,j = 1,\ldots, N+M,\ i \neq j \qquad (9)$$

$X_{ijk} = 0,1$ for all i, j, k, U_i are positive integers for all i where

$$X_{ijk} = \begin{cases} 1 & \text{if link } i,j \text{ is included in tour } k, \\ 0 & \text{otherwise} \end{cases}$$

C_{ij} represents the traveling distance from location i to location j;
$T(L)$ represents the traveling distance for tour L;
T represents the total distance traveled in the maximal tour;
D represents the maximal difference among allocation of distance traveled for each tour;
U_i represents a vector of positive integer values which gives a set of subtour elimination constraints in constraint (9);
M represents the number of tours or salesmen or vehicles; and
N represents the number of locations to be visited.

Also, (1),(2),(3) denote objective functions of minimizing the maximal tour, minimizing the total distance and minimizing the largest deviation respectively; (4),(5) ensure that T and D represent the maximal tour and the maximal deviation respectively; (6),(7) ensure that each point is visited by one and only one vehicle; (8) represents route continuity; and (9) represents subtour elimination constraints.

Due to the combinatorial nature and complexity of multiple objectives in the above model, there is no computationally feasible exact procedure available. A goal program may be