Uniformly Most Powerful tests of Two-Sided Hypotheses for Geometric Distribution

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1. Introduction

Testing of statistical hypotheses is one of the important problems in statistical inference for making decision as to reject or accept the test hypothesis. The decision is made based on the values of a random variable X, the distribution P_θ of which is belong to a class $\mathbf{P} = \{P_\theta, \theta \in \Omega\}$. The distributions in **P** can be classified into those of which the hypothesis is true, H and those for which it is false, K. The resulting two classes are $H \cap K = \phi$ and $H \cup K = \mathbf{P}$, and the corresponding subsets $\Omega_H = \{\theta : \theta \in H\}$, $\Omega_K = \{\theta : \theta \in K\}$ are $\Omega_H \cap \Omega_K = \phi$ and $\Omega_H \cup \Omega_K = \Omega$ respectively.

For the best hypothesis testing, one should select the test φ so as to maximize the power $\beta_\varphi(\theta) = E_\theta\, \varphi(X)$ for all $\theta \in \Omega_K$ subject to the condition $E_\theta\, \varphi(X) \leq \alpha$ for all $\theta \in \Omega_H$.

There exists a uniformly most powerful test for testing hypothesis $H : \theta \in \omega$ against alternative $K : \theta \in \Omega - \omega$ with size of the test α if and only if $\sup\beta_\varphi(\theta) = \sup E_\theta\, \varphi(X) = \alpha$ for all $\theta \in \omega$ and $\beta_\varphi(\theta) = E_\theta\, \phi(X) \geq E_\theta\, \varphi^*(X) = \beta_\varphi{*}(\theta)$ for all $\theta \in \Omega - \omega$ where φ^* be any test that $\sup\beta_\varphi{*}(\theta) = \sup E_\theta\, \varphi^*(X) \leq \alpha$ for all $\theta \in \omega$ (Lehmann, 1986: 79)

There are three types of two-sided hypotheses:

Type 1 : $H_1 : \theta \leq \theta_1$ or $\theta \geq \theta_2$ against $K_1 : \theta_1 < \theta < \theta_2$

Type 2 : $H_2 : \theta_1 \leq \theta \leq \theta_2$ against $K_2 : \theta < \theta_1$ or $\theta > \theta_2$

Type 3 : $H_3 : \theta = \theta_0$ against $K_3 : \theta \neq \theta_0$

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There exists a uniformly most powerful test for the hypothesis of type1 and a uniformly most powerful unbiased test for type 2 and 3. (Lehmann, 1986: 101-2, 135). The latter is said to be uniformly most powerful test in the class of all unbiased tests. For this study it concentrate only the hypothesis of type 1 for uniformly most powerful test. In practice, uniformly most powerful test does not exist for all situations but it does for some. It is not easy to find this in the real problem. Thus, the objectives of this study is to state the conditions for the existing of a uniformly most powerful test for testing the two-sided hypothesis in Geometric distribution population.

2. Scope of study

There are many factors involve in a uniformly most powerful test. In this study we consider only the following cases.

1. Hypothesis of interest

   $H_1 : \theta \leq \theta_1$ or $\theta \geq \theta_2$ against $K_1 : \theta_1 < \theta < \theta_2$

2. Distribution of random variable

   Let X be a random sample from geometric population with parameter θ and the probability density function is $P_\theta(X = x) = \theta(1- \theta)^{x-1}$, $x = 1, 2, \ldots$; $0 \leq \theta \leq 1$. If $X_1, X_2, \ldots, X_m$ be a random sample from geometric population,

   then $Y = \sum_{i=1}^{m} X$ have negative binomial distribution (Rohatgi, 1976: 189) where Y is the number of trials until the m^{th} success occurred and the probability density function of Y is

   $$P_\theta(Y = y) = \binom{y-1}{m-1}\theta^m (1-\theta)^{y-m}, \quad y = m, m+1, m+2, \ldots ; 0 \leq \theta \leq 1$$

   where θ is the probability of success occurred in each trial.

3. Number of successes m to be consider are : 10, 20, 30, 40, and 50

4. Size of the test α are 0.01 and 0.05.

3. Method of study

Lehmann (1986) stated that for testing the hypothesis $H : \theta \le \theta_1$ or $\theta \ge \theta_2$ against the alternatives $K : \theta_1 < \theta < \theta_2$ in the one-parameter exponential family

$$P_\theta(x) = C(\theta)e^{Q(\theta)T(x)}.h(x)$$

where $Q(\theta)$ is monotone function of θ then there exists a uniformly most powerful test.

If $Q(\theta)$ is monotone increasing function of θ, a uniformly most powerful test is given by

$$\varphi(x) = \begin{cases} 1 & \text{when} \quad c_1 < T(x) < c_2 \\ \gamma_i & \text{when} \quad T(x) = c_i,\ i = 1,2 \\ 0 & \text{when} \quad T(x) < c_1 \text{ or } T(x) > c_2 \end{cases}$$

where c_i and γ_i are determined by $E_{\theta 1}\ \varphi(X) = E_{\theta 2}\ \varphi(X) = \alpha$

If $Q(\theta)$ is monotone decreasing function of θ, a uniformly most powerful test is given by

$$\varphi(x) = \begin{cases} 1 & \text{when} \quad T(x) < c_1 \text{ or } T(x) > c_2 \\ \gamma_i & \text{when} \quad T(x) = c_i,\ i = 1,2 \\ 0 & \text{when} \quad c_1 < T(x) < c_2 \end{cases}$$

where c_i and γ_i are determined by $E_{\theta 1}\ \varphi(X) = E_{\theta 2}\ \varphi(X) = \alpha$

Let Y be a negative binomial random variable, the probability density function is

$$P_\theta(Y = y) = \binom{y-1}{m-1}\theta^m(1-\theta)^{y-m} \quad , \ y = m, m+1, m+2, \ldots \ ; 0 \le \theta \le 1$$

$$= \left(\frac{\theta}{1-\theta}\right)^m e^{y\ln(1-\theta)}.\binom{y-1}{m-1}$$

$$= C(\theta)e^{Q(\theta)T(y)}.h(y)$$

where $C(\theta) = \left(\frac{\theta}{1-\theta}\right)^m$, $Q(\theta) = \ln(1-\theta)$, $T(y) = y, h(y) = \binom{y-1}{m-1}$ does not depend on θ and $Q(\theta) = \ln(1-\theta)$ is a decreasing function of θ. A uniformly most powerful test for $H_1 : \theta \le \theta_1$ or $\theta \ge \theta_2$ against the alternatives $K_1 : \theta_1 < \theta < \theta_2$ with size α is given by

$$\varphi(x) = \begin{cases} 1 & \text{where } T(x) < c_1 \text{ or } T(x) > c_2 \\ \gamma_i & \text{where } T(x) = c_i,\ i = 1,2 \\ 0 & \text{where } c_1 < T(x) < c_2 \end{cases}$$

where c_i and γ_i are determined by $E_{\theta 1}\ \varphi(X) = E_{\theta 2}\ \varphi(X) = \alpha$ or by the following equations

$$\sum_{y=m}^{c_1-1} P_{\theta_1}(y) + \sum_{y=c_2+1}^{\infty} P_{\theta_1}(y) + \gamma_1 P_{\theta_1}(c_1) + \gamma_2 P_{\theta_1}(c_2) = \alpha \qquad (1)$$

and

$$\sum_{y=m}^{c_1-1} P_{\theta_2}(y) + \sum_{y=c_2+1}^{\infty} P_{\theta_2}(y) + \gamma_1 P_{\theta_2}(c_1) + \gamma_2 P_{\theta_2}(c_2) = \alpha \qquad (2)$$

For the given values of θ_1, θ_2, m and α, we want to find the value of c_1 and c_2 that satisfy the two equations and $0 < \gamma_1 < 1$, $0 < \gamma_2 < 1$. The trials are given by the following procedures:

1. Let m = 10
2. Find the probability distribution and cumulative probability distribution for both directions of θ start from 0.1 to 0.9 with increment of 0.1
3. Given the values of all possible pairs of θ_1 and θ_2 and $\alpha = 0.01$ or 0.05, then try to find the values of c_1 and c_2 with the two constraints (1) and (2)
4. Change the values of m to 20, 30, 40, 50 and repeat step 2 and 3.

4. Results

It can be seen from the result of trials shown in table 1 that we can not find a uniformly most powerful test in every paired values of θ_1 and θ_2. The pattern of the probability distribution of Y is change by the value of θ, so we can not find a uniformly most powerful test when the different values of θ_1 and θ_2 is greater than 0.2. For θ_1 or θ_2 is equal to 0.1 or 0.2, the pattern of the probability distribution of Y is flat with long tailed on both sides so that it is not possible to find the test.

Table 1. Paired of θ which have a uniformly most powerful test.

(θ_1,θ_2)	(0.3, 0.4)	(0.3,0.5)	(0.4,0.5)	(0.4,0.6)	(0.5,0.6)	(0.5,0.7)	(0.6,0.7)	(0.7,0.8)	(0.8,0.9)
m=10	√	√*	√	√	√	√*	√*		
m=20			√		√		√		
m=30					√		√	√	
m=40					√		√	√	
m=50							√	√	

* having a uniformly most powerful test only at $\alpha = 0.05$

We found that at $\theta_1 = 0.6$ and $\theta_2 = 0.7$, there exits a uniformly most powerful test of size $\alpha = 0.05$ for m =10 to m =50. The value of c_1 and c_2 are showed in table 2.

Table 2. The values of c_1 and c_2 for testing hypothesis $H_1 : \theta \le 0.6$ or $\theta \ge 0.7$ against the alternatives $K_1 : 0.6 < \theta < 0.7$

α	m	γ_1	c_1	γ_2	c_2
0.01	10	--	--	--	--
	20	0.28828386	22	0.33844385	46
	30	0.77505223	34	0.29882101	65
	40	0.48866713	47	0.58607233	84
	50	0.41832076	60	0.12190431	102
0.05	10	0.22743176	11	0.17491338	23
	20	0.85348905	23	0.49960879	42
	30	0.85028461	36	0.22021005	60
	40	0.06014867	50	0.15127516	78
	50	0.35546511	63	0.21792436	96

Thus for practical situations, we can state the following three additional conditions where a uniformly most powerful test exists.

1. $P_{\theta_2}(Y = m) < \alpha$

2. $0 < P_{\theta_2}(Y < c_1) < \alpha, P_{\theta_2}(Y \leq c_1) \geq \alpha$ and

$0 < P_{\theta_1}(Y > c_2) < \alpha, P_{\theta_1}(Y \geq c_2) \geq \alpha$

3. $\frac{\alpha}{2} < P_{\theta_2}(Y < c_1) + P_{\theta_2}(Y > c_2) < \alpha$ and

$\frac{\alpha}{2} < P_{\theta_1}(Y < c_1) + P_{\theta_1}(Y > c_2) < \alpha$

5. Conclusion

For testing two sided hypotheses of $H_1 : \theta \leq \theta_1$ or $\theta \geq \theta_2$ against the alternatives $K_1 : \theta_1 < \theta < \theta_2$, it was shown by Lehman that there is a uniformly most powerful test. But for practice it is not easy to find the UMP test, there are many involved factors such as the population distribution, the values of interesting parameters, size of the test, and the number of trials or the number of successes. In the case of negative binomial distribution, we set the number of successes to study at m=10, 20, 30, 40, and 50.

We found a uniformly most powerful test in some situations as shown in table 1 and the three additional conditions for finding the uniformly most powerful test.

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