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Powers of Some One-Sided Multivariate Tests with the Population Covariance Matrix Known up to a Multiplicative Constant

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ประกาศสถาบันบัณฑิตพัฒนบริหารศาสตร์
เรื่อง ผลการคัดเลือกบทความวิชาการดีและบทความวิชาการดีเด่น ประจำปี 2547

ตามประกาศสถาบันบัณฑิตพัฒนบริหารศาสตร์ ลงวันที่ 15 มิถุนายน 2547 ได้ประกาศเชิญชวนให้ข้าราชการและพนักงานของสถาบันส่งบทความวิชาการเข้ารับการพิจารณาคัดเลือกเป็นบทความวิชาการดีและบทความวิชาการดีเด่น ประจำปี 2547 ใน 11 สาขาวิชา คือ สาขาวิชารัฐประศาสนศาสตร์ บริหารธุรกิจ พัฒนาการเศรษฐกิจ สถิติประยุกต์ คอมพิวเตอร์และสารสนเทศ พัฒนาสังคม พัฒนาทรัพยากรมนุษย์ เทคโนโลยีการบริหาร ภาษาและการสื่อสาร การจัดการสิ่งแวดล้อม และสาขาวิชาสังคมศาสตร์อื่นๆ โดยบทความวิชาการที่ได้รับการคัดเลือก จะได้รับเงินรางวัล ดังนี้

1. บทความวิชาการดีเด่น ได้รับเงินรางวัล บทความละ 50,000.- บาท
2. บทความวิชาการดี ได้รับเงินรางวัล บทความละ 30,000.- บาท
3. บทความวิชาการชมเชย ได้รับเงินรางวัล บทความละ 10,000.- บาท

สถาบันบัณฑิตพัฒนบริหารศาสตร์ได้รับบทความวิชาการที่ส่งเข้ารับการคัดเลือกเป็นบทความวิชาการดีและบทความวิชาการดีเด่น จำนวนทั้งสิ้น 6 บทความ เป็นบทความในสาขาวิชารัฐประศาสนศาสตร์ 1 บทความ พัฒนาการเศรษฐกิจ 1 บทความ สถิติประยุกต์ 1 บทความ คอมพิวเตอร์และสารสนเทศ 2 บทความ พัฒนาองค์การ 1 บทความ ซึ่งคณะกรรมการดำเนินงานคัดเลือกบทความวิชาการดีและบทความวิชาการดีเด่นได้พิจารณาบทความดังกล่าวเสร็จเรียบร้อยแล้ว ผลปรากฏว่าบทความวิชาการในสาขาวิชาดังต่อไปนี้ เป็นบทความวิชาการที่สมควรได้รับรางวัลชมเชย

1. บทความสาขาวิชาพัฒนาการเศรษฐกิจ เรื่อง "แร่ตะกั่วที่ห้วยคลิตี้ จังหวัดกาญจนบุรี" ของ ผู้ช่วยศาสตราจารย์ อดิศร์ อิศรางกูร ณ อยุธยา
2. บทความสาขาวิชาสถิติประยุกต์ เรื่อง "Powers of Some One-Sided Multivariate Tests with the Population Covariance Matrix Known up to a Multiplicative Constant" ของ รองศาสตราจารย์ สำรวม จงเจริญ
3. บทความสาขาวิชาคอมพิวเตอร์และสารสนเทศ เรื่อง "Combining Prediction by Partial Matching and Logistic Regression for Thai Word Segmentation" ของ อาจารย์ โอม ศรนิล
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รองศาสตราจารย์

(ปรีชา จรุงกิจอนันต์)

อธิการบดีสถาบันบัณฑิตพัฒนบริหารศาสตร์

Powers of Some One-Sided Multivariate Tests with the Population Covariance Matrix Known up to a Multiplicative Constant

โดย

รองศาสตราจารย์ สำรวม จงเจริญ

Powers of Some One-Sided Multivariate Tests with the Population Covariance Matrix Known up to a Multiplicative Constant

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Abstract

For a multivariate normal population, Kudo (Biometrika 50 (1963) 403) and Shorack (Ann. Math. Statist. 38 (1967) 1740) derived the likelihood ratio tests of the null hypothesis that the mean vector is zero with a one-sided alternative for a known covariance matrix and for a covariance matrix which is known up to a multiplicative constant, respectively. Because these tests may be tedious to use, Tang et al. (Biometrika 76 (1989) 577) developed an approximate likelihood ratio test and Follmann (J. Amer. Statist. Assoc. 91 (1996) 854) proposed a one-sided modi7cation of the usual chi-squared test for an unordered alternative. We consider a modification of Follmann's test which performs better than Follmann's test at some alternatives, and we derive expressions for the powers of the new test and Follmann's test for the cases considered here. For multivariate normal distributions with dimension no more than three and known covariance matrix, we consider the power functions of Kudo's test and the Tang–Gnecco–Geller test. Using these exact results and Monte-Carlo simulations, we study the powers of these one-sided tests.

MSC: primary 62H15; secondary 62F03

Keywords: Follmann's test; Kudo's test; Shorack's test; Simple order; Simple order; Tang–Gnecco–Geller test

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1. Introduction and summary

Suppose one uses a matched pair design to compare the multivariate responses of two treatments. If the responses are p-dimensional and $\theta = (\theta_1, \theta_2, \ldots, \theta_p)'$ is the difference, treatment one minus treatment two, of the mean responses, then one may test the null hypothesis, H_0: $\theta_1 = \theta_2 = \cdots = \theta_p = 0$, to determine if there is a difference in the two treatments. Furthermore, if one believes that for each coordinate, the mean responses for treatment one are at least as large as those for treatment two, then the alternative can be constrained by H_1: $\theta_i \geq 0$ for $i = 1, 2, \ldots, p$.

Based on a random sample from the normal distribution with mean θ and covariance matrix V, Kudo (1963) and Shorack (1967) derived the likelihood ratio tests of H_0 versus H_1-H_0 for the cases in which V is known and V is known up to a multiplicative constant, respectively. Because the likelihood ratio tests with restricted alternatives are complicated to use, Tang et al. (1989) proposed an approximate likelihood ratio test and Follmann (1996) proposed one-sided modifications of the usual χ^2 and Hotelling's T^2 tests of H_0 versus ~H_0 which are easier to implement. We extend Follmann's test and the Tang–Gnecco–Geller test to the setting in which the covariance matrix is known up to a multiplicative constant, and we study the power of these one-sided tests. Powers of one-sided tests for V completely unknown are considered elsewhere.

These hypotheses also arise in the one-way analysis of variance when the means are known to satisfy an order restriction. For observations which come from k normal populations whose means are known to satisfy a simple ordering, i.e. $H_S : \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_k$, Bartholomew (1959a, b, 1961) derived the likelihood ratio test of $\mu_1 = \mu_2 = \cdots = \mu_k$ with the alternative restricted by H_S for the cases of known variances and variances known up to a multiplicative constant. Suppose the observations are Y_{ij} for $j = 1, 2, \ldots, n_i$ and $i = 1, 2, \ldots, k$ and the sample means are $\bar{Y}_1, \bar{Y}_2, \ldots, \bar{Y}_k$. With variances, σ_1^2, $\sigma_2^2, \ldots, \sigma_k^2$, known, Kudo (1963) noted that for $p = k - 1$, $X_i = \bar{Y}_{i+1} \;\; \bar{Y}_i$ for $i = 1, 2, \ldots, p$, $X = (X_1, X_2, \ldots, X_p)'$ and $\theta = E(X)$, the hypotheses on μ are equivalent to H_0 and H_1 above, and Bartholomew's and Kudo's tests are equivalent. With $w_i = n_i/\sigma_i^2$ for $i = 1, 2, \ldots, k$, the correlation matrix for X satisfies

$$\rho_{i,i+1} = -\sqrt{\frac{w_i w_{i+2}}{(w_i + w_{i+1})(w_{i+1} + w_{i+2})}} \quad \text{for } i = 1, 2, \ldots, p-1 \tag{1.1}$$

and

$$\rho_{ij} = 0 \text{ for } |i - j| \geqslant 2.$$

If the variances are known up to a multiplicative constant and $n_1 = n_2 = \cdots = n_k = n$, then one could take differences $X_{ij} = Y_{i+1,j} - Y_{i,j}$ for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, p$. Thus, Shorack's test can be used to test homogeneity of the means with a simply ordered alternative.

Actually Bartholomew considered an arbitrary partial order restriction which includes the simple tree order, i.e. $H_T : \mu_1 \leqslant \mu_j$ for $j = 2, 3, \ldots, k$. For this ordering, one takes differences $X_i = \bar{Y}_{i+1} - \bar{Y}_1$ or $X_{ij} = Y_{i+1,j} - Y_{1,j}$ for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, p$, and with w_i as above, the correlation matrix of $X = (X_1, X_2, \ldots, X_p)'$ satisfies

$$\rho_{ij+1} = \sqrt{\frac{w_{i+1} w_{j+1}}{(w_{i+1} + w_1)(w_{j+1} + w_1)}} \quad \text{for } 1 \leqslant i \neq j \leqslant p. \tag{1.2}$$

In Section 3, we compare the powers of Shorack's and Bartholomew's tests for the simple order and the simple tree order.

Robertson et al. (1988, Section 4.2) discuss the use of contrast tests for the alternatives H_S and H_T. Also, for the alternative H_T, Tang and Lin (1997) apply the techniques of Tang et al. (1989), and Conaway et al. (1991) consider the use of circular tests.

For V known with p = 2 and 3, we obtain formulas for the power of Kudo's test and show how the power of the Tang–Gnecco–Geller test can be obtained from these formulas. While expressions for the power of these tests can be obtained for larger p and for V known up to a multiplicative constant, this was not done because the expressions are quite complicated. We use Monte-Carlo techniques to study the power of these tests in such cases. Expressions for the power function of Follmann's test and its modification are derived for arbitrary p and V known or V known up to a multiplicative constant.

In the power study, for a known covariance matrix V or $V = \sigma^2 V_0$, which is discussed in Sections 2 and 3, we considered V or V_0 determined by (1.1) and (1.2) as well as several other covariance structures. For V known, we considered Kudo's test, Follmann's test, the new test, which is a modification of Follmann's test, and the Tang–Gnecco–Geller test. For $V = \sigma^2 V_0$, we considered the same tests with Shorack's test in place of Kudo's test and Bartholomew's test included for the covariance matrices (1.1) and (1.2).

For V known, Kudo's test has the best overall powers of the four tests. If V has few positive correlations, then one could use the Tang–Gnecco–Geller test which has larger power than Kudo's test at the center of H_1, but substantially smaller powers at some of its edges. If V has several positive correlations, Follmann's test, the new test and the Tang–Gnecco–Geller test are not recommended because they have smaller powers than the usual chi-square test at some of the edges of H_1.

For $V = \sigma^2 V_0$ with V_0 of the form (1.1) or (1.2), Bartholomew's test should be used. However, if $n \geqslant 20$, there is little difference in the powers of Bartholomew's and Shorack's tests. For the other covariance matrices considered, the conclusions are like those for known V. In particular, Shorack's test has the best overall powers. If V has few positive correlations, one could use the Tang–Gnecco–Geller test which has larger power than Shorack's test at the center of H_1, but substantially smaller powers at some of its edges. If V has several positive correlations, Follmann's test, the new test and the Tang–Gnecco–Geller test should not be used if the alternative may be near one of the edges of H_1.

These results are not surprising. Kudo's and Shorack's tests are the likelihood ratio tests of the hypotheses of interest, and the Tang–Gnecco–Geller test is the likelihood ratio test of H_0 with H_1 changed to $\theta \varepsilon A\Re_+^p$ where A is a $p \times p$, non-singular matrix which depends on V. As we shall see, for some V, some of the edges of $\Re_+^p$ can be significantly far away from $A\Re_+^p$, and the Tang–Gnecco–Geller test does not perform well at such edges.

Table 1

Powers times 1000 of the tests for known V with the correlation matrices (1.1) and (1.2), i.e. the simple and simple tree orders with equal weights and p = 2 and 3

Simple order, p = 2					Simple tree, p = 2			
Test					Test			
Direction	π_K	π_F	π_N	π_T	π_K	π_F	π_N	π_T
(1, 1)	846	800	800	824	801	800	800	824
(0, 1)	823	798	798	815	770	747	747	744
(1, 0)	823	798	798	815	770	747	747	744
Simple order, p = 3					**Simple tree, p = 3**			
Test					Test			
Direction	π_K	π_F	π_N	π_T	π_K	π_F	π_N	π_T
(1, 1, 1)	878	799	800	843	804	800	800	843
(1, 0, 1)	868	800	800	842	776	775	775	751
(0, 1, 1)	866	798	799	837	776	775	775	751
(1, 1, 0)	866	798	799	837	776	775	775	751
(1, 0, 0)	840	797	797	826	757	682	683	724
(0, 1, 0)	848	791	798	832	757	682	683	724
(0, 0, 1)	840	796	797	826	757	682	683	724

Follmann's test focuses its power on the half space $\{\theta : \theta_1 + \cdots + \theta_p \geqslant 0\}$, and it does not reject H_0 if the sample mean vector is outside of this half space. Because $\Re^p_+$ is contained in this half space, the likelihood ratio tests have larger powers than Follman's test. When the angles, which are determined by the inner product $(x, y) = x' V^{-1} y$, are large then these differences in power are less noticeable in the "middle" of $\Re^p_+$. For instance, with e_i the p-dimensional vector with ith coordinate 1 and the other coodinates 0, $V_{ii} = 1$ and $V_{ij} = 0.5$ for $1 \leqslant i \neq j \leqslant p$, the cosine of the angle between e_i and e_j is $e_i' V^{-1} e_j / \sqrt{(e_j' V^{-1} e_j)(e_j' V^{-1} e_j)}$ which is $-\frac{1}{3}$ for $p = 3$ and $1 \leqslant i \neq j \leqslant 3$, and the associated angle is 109.5°. We see from Tables 1 and 4, the powers of the likelihood ratio test and Follmann's test are close in the middle of $\Re^p_+$ for this V.

However, for this V the edges, e_i, are close to the boundary of the half space. Let $d_i = (d_{i1}, d_{i2}, \ldots, d_{ip})'$ with $d_{ii} = 1$ and $d_{ij} = -1/(p-1)$ for $1 \leqslant i \neq j \leqslant p$. For $p = 3$, the angle between e_i and d_i is 30°, and is 9.6° for $p = 6$. Because Follmann's test does not reject if the sample mean vector is outside of the half space, its power is bounded above by $\frac{1}{2}$ for any θ on the boundary of the half space. Because d_i is on the boundary of the half space and e_i is close to d_i for this V, this explains the poor performance for Follmann's test at the edges, e_i, for this V.

Similar comments hold for the new test presented in Section 2.4, which is a modification of Follmann's test. In Section 2.4, we also give a simple method for choosing between Follmann's test and the modification presented here.

2. One-sided tests with known covariance matrix

Let $X_1, X_2, \ldots, X_n$ be a random sample from a multivariate normal distribution with mean $\theta = (\theta_1, \theta_2, \ldots, \theta_p)'$ and known covariance matrix V. We consider the problem of testing $H_0 : \theta = 0$ versus $H_1 - H_0$ where $H_1 : \theta \in \Re^p_+$ and $\Re^p_+ = \{x : x_i \geqslant 0 \text{ for } i = 1, 2, \ldots, p\}$ is the non-negative orthant in $\Re^p$.

2.1 Kudo's test

From Robertson et al. (1988, pp. 219–220), the likelihood ratio test (LRT) of H_0 versus $H_1 - H_0$ rejects H_0 for large values of

$$\bar{\chi}^2_{01} = n\theta^{*\prime} V^{-1} \theta^*, \tag{2.1}$$

where θ^*, which minimizes $(\bar{X} - \theta)' V^{-1} (\bar{X} - \theta)$ subject to $\theta \in \Re^p_+$, can be computed using a quadratic programming routine such as QPROG in IMSL. The null hypothesis distribution of $\bar{\chi}^2_{01}$ is given by their Theorem 4.6.1, i.e. for any real number t,

$$P(\bar{\chi}^2_{01} \geqslant t) = \sum_{j=0}^{p} Q(j, p; V) P(\chi^2_j \geqslant t), \tag{2.2}$$

where χ^2_j is a chi-squared variable with j degrees of freedom ($\chi^2_0 \equiv 0$) and the weights $Q(j, p; V)$, $j = 0, 1, 2, \ldots, p$, are nonnegative, sum to one and can be computed using the FORTRAN programs by Bohrer and Chow (1978) and Sun (1988) for $p < 10$.

Next, we derive the power function of Kudo's test with critical value $c > 0$. For details see Chongcharoen (1998). For simplicity we take $n = 1$ and $\sigma_{ii} = 1$ for all $i = 1, 2, \ldots, p$. For arbitrary n and σ_{ii} the formula below can be used if each θ_i is multiplied by $\sqrt{n/\sigma_{ii}}$.

With ϕ and Φ the density and distribution function of the standard normal distribution and $\theta = (\theta_1, \theta_2, \cdots, \theta_p)'$, define

$$P(1, \theta_1, c) = \Phi(\theta_1 - \sqrt{c}) \tag{2.3}$$

and

$$P(2, \theta_1, \theta_2, p, c) = \frac{e^{-(1/2)\Gamma^2}}{2\pi} \int_{\alpha-\pi/2+\beta}^{\pi-\alpha+\beta} \Psi_1(\Gamma \sin\gamma, \sqrt{c})\, d\gamma, \tag{2.4}$$

where

$$U = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

$\Gamma = \sqrt{\theta' U^{-1}\theta}$, α is determined by $\cos\alpha = (\sqrt{1+\rho} - \sqrt{1-\rho})/2$ with $0 \leqslant \alpha \leqslant \pi$, β is determined by $U^{-(1/2)}\theta = \Delta\,(\sin\beta, \cos\beta)'$ and $\Psi_1(a, b) = [\phi\,(a-b) + a\Phi\,(a-b)]\,/\,\phi\,(a)$.

Also define

$$P\,(3, \theta_1, \theta_2, \theta_3, \rho_{12}, \rho_{13}, \rho_{23}, c)$$

$$= \frac{e^{-(1/2)\tau'\tau}}{(2\pi)^{3/2}} \int_0^{\pi}\int_0^{2\pi} \sin\gamma\, I(\gamma, \varphi)\ \Psi_2\,(\Lambda, \sqrt{c}\,)\, d\varphi\, d\gamma, \tag{2.5}$$

where W is a 3×3 matrix with $W_{ii} = 1$ for $1 \leqslant i \leqslant 3$, $W_{ji} = W_{ij} = \rho_{ij}$ for $1 \leqslant i < j \leqslant 3$, $\tau = W^{-(1/2)}\,\theta$, $I\,(\gamma, \phi)$ is the indicator of $\{(\gamma, \phi) : \cos\gamma > 0, \sin\gamma\cos\phi > 0 \text{ and } \sin\gamma\sin\phi > 0\}$, $\Psi_2(a, b) = [(a+b)\phi\,(a-b) + (1+a^2)\,\Phi\,(a-b)]/\phi\,(a)$, and $\Lambda = \tau_1\cos\gamma + \tau_2\sin\gamma\cos\phi + \tau_3\sin\gamma\sin\phi$.

For $p = 2$, with $\rho = V_{12}$ and π_K the power of Kudo's test,

$$\pi_K(\theta) = P\,(2, \theta_1, \theta_2, \rho, c) + \Phi\,(-\theta_2)\Phi\left[\frac{\theta_1 - \rho\theta_2}{\sqrt{1-\rho^2}} - \sqrt{c}\right]$$

$$+\ \Phi\,(-\theta_1)\Phi\left[\frac{\theta_2 - \rho\theta_1}{\sqrt{1-\rho^2}} - \sqrt{c}\right]. \tag{2.6}$$

It is instructive to note that the second term in (2.6) is

$$P\,(1, -\theta_2, 0)P\left(1, \frac{\theta_1 - \rho\theta_2}{\sqrt{1-\rho^2}}, c\right)$$

and that the third term can be obtained from the second by interchanging θ_1 and θ_2.

For $p = 3$, with $\rho_{ij} = V_{ij}$

$$\pi_K(\theta) = P(3, \theta_1, \theta_2, \theta_3, V_{12}, V_{13}, V_{23}, c) + a_{12} + a_{13} + a_{23} + b_1 + b_2 + b_3, \tag{2.7}$$

where $a_{12} = P(1, \bar\theta_3\,(\theta, V), 0)\ P\,(2, \bar\theta_1(\theta, V), \bar\theta_2(\theta, V), \bar\rho(V), c)$,

$$\bar\theta_1(\theta, V) = \frac{\theta_1 - V_{13}\theta_3}{\sqrt{1-V_{13}^2}}, \quad \bar\theta_2(\theta, V) = \frac{\theta_2 - V_{23}\theta_3}{\sqrt{1-V_{23}^2}}, \quad \bar\theta_3\,(\theta, V) = \theta_3,$$

$$\bar\rho(V) = \frac{V_{12} - V_{13}V_{23}}{\sqrt{(1-V_{13}^2)(1-V_{23}^2)}}, \quad L_{132} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_{321} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and a_{13} and a_{23} are computed as a_{12} with (θ, V) replaced by $(L_{132}\theta, L_{132}VL_{132}')$ and $(L_{321}\theta, L_{321}VL_{321}')$, respectively. Also

$$b_1 = P(1, \tilde\theta_1\,(\theta, V), c)\ P\,(2, \tilde\theta_2\,(\theta, V), \tilde\theta_3\,(\theta, V), \tilde\rho\,(V), 0),$$

where

$$\tilde{\theta}_1(\theta, V) = \left[\frac{1 - V_{23}^2}{1 - V_{12}^2 - V_{13}^2 - V_{23}^2 + 2V_{12}V_{13}V_{23}}\right]^{1/2}$$

$$\times \left\{\theta_1 - \frac{(V_{12} - V_{13}V_{23})\theta_2 - (V_{13} - V_{12}V_{23})\theta_3}{1 - V_{23}^2}\right\},$$

$$\tilde{\theta}_2(\theta, V) = \frac{V_{23}\theta_3 - \theta_2}{\sqrt{1 - V_{23}^2}}, \quad \tilde{\theta}_3(\theta, V) = \frac{V_{23}\theta_2 - \theta_3}{\sqrt{1 - V_{23}^2}},$$

$$\tilde{\beta}(V) = -V_{23}, \; L_{132} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and b_2 and b_3 are computed as b_1 with (θ, V) replaced by $(L_{213}\theta, L_{213}VL_{213}')$ and $(L_{321}\theta, L_{321}VL_{321}')$, respectively.

When $p \geq 4$, the power function of $\bar{\chi}^2_{01}$ is more complex.

2.2 An approximate likelihood ratio test

Tang et al. (1989) proposed an approximate likelihood ratio test. With $V^{-1/2}$ symmetric, let $Z = \sqrt{n}V^{-1/2}\bar{\chi} \sim N(\eta, I)$ where $\eta = \sqrt{n}V^{-1/2}\theta$. H_0 is rejected for large values of

$$T = \sum_{j=1}^{p} (\max(Z_j, 0))^2$$

and under H_0, for any real number t,

$$P(T \geq t) = \sum_{j=0}^{p} \frac{p!}{j!(p-j)!} P(\chi_j^2 \geq t)/2^p.$$

They give the critical values of their test for $p \leq 10$.

We now relate the power of T, π_T, to the power of Kudo's test. Consider Kudo's test of $\eta_1 = \eta_2 = \cdots = \eta_p = 0$ versus $\eta_i \geq 0$ for $i = 1, 2, \ldots, p$ with at least one strict inequality. If one bases the test on Z, since the covariance matrix is the identity, the projection of Z onto the orthant, η^*, minimizes $\sum_{j=1}^{p}(Z_j - \eta_j)^2$ subject to $\eta_j \geq 0$ for $i = 1, 2, \ldots, p$. Clearly, this minimization problem can be solved coordinate-wise. Thus, $\eta_j^* = \max(Z_j, 0)$ for $j = 1, 2, \ldots, p$, Kudo's test rejects for large values of T, and $\pi_T = \pi_K$ with θ and V replaced by $\eta = \sqrt{n}V^{-(1/2)}\theta$ and the identity matrix.

The null distribution of T is valid if $V^{-(1/2)}$ is replaced by any matrix A as long as the covariance of $\sqrt{n}A\bar{X}$ is the identity. Tang et al. (1989) derived this approximate likelihood ratio test by replacing $A\Re_+^p$ by $\Re_+^p$. They show how such an A can be computed so that a "center" of $A\Re_+^p$ is the ray dJ with $d > 0$ and J the p-dimensional vector with each coordinate 1. In all the cases considered in our study, the largest gain in power resulting from the use of their A rather than $V^{-1/2}$ was 0.03(3.8%) which occurred with $p = 6$, the simple tree covariance, $\theta = d\,(0, 1, 1, 1, 1, 1)'$ and $d > 0$. However, using A rather than $V^{-1/2}$ resulted in a loss of power as large as 0.23(38.3%) which occurred for $p = 6$, the last covariance matrix in Table 3 (a mixture of positive and negative correlations), $\theta = d\,(0, 0, 0, 0, 1, 1)'$ and $d > 0$.

The matrix A was chosen so that the "center" of $A\Re_+^p$ and $\Re_+^p$ are aligned, and the test based on A performs slightly better than that based on $V^{-1/2}$ for alternatives near that center. To understand its poor performance near the edges of $\Re_+^p$ in some cases, we consider Ae_i and $V^{-1/2}e_i$ for the p unit vectors, e_i, and the covariance matrix in Table 3 for which the test based on A performed so poorly. Of the edges $V^{-1/2}e_i$, the one closest to e_i is $V^{-1/2}e_i$ and the angles between $V^{-1/2}e_i$ and e_i range from 42.9° to 48.4°. Of the edges Ae_i, the one closest to e_i is Ae_i and the angles between Ae_i and e_i range from 24.0° to 79.6°. Thus, the "center" of $A\Re_+^p$ agrees with the center of $\Re_+^p$, but some of its edges are far from the edges of

$\Re^p_+$. This is one explanation for the poor performance of the test based on A near some of the edges in this case. We recommend $V^{-1/2}$ over A and do not consider the version based on A further.

2.3 Follmann's test

With $X^2 = n\bar{X}' V^{-1}\bar{X}$, Follmann's (1996) test (denoted χ^2_+) rejects H_0 in favor of $H_1 - H_0$ at level α if

$$X^2 > \chi^2_{2\alpha,p} \text{ and } \sum_{j=1}^{p} \bar{X}_j > 0, \tag{2.8}$$

where $\chi^2_{2\alpha,p}$ is the $1 - 2\alpha$th quantile of the central chi-squared distribution with p degrees of freedom.

We now derive an expression for the power of Follmann's test. With $V^{-1/2}$ symmetric, J the p-dimensional vector with $J_i = 1$ for $1 \leq i \leq p$, $Z = \sqrt{n}V^{-1/2}\bar{X} \sim N(\eta, I)$ where $\eta = \sqrt{n}V^{-1/2}\theta$, and $L = J'V^{1/2}Z/\sqrt{J'VJ}$, $X^2 = Z'Z$ and $\bar{X}_1 + \cdots + \bar{X}_p > 0$ if and only if $L > 0$. Because

$$C = I - \frac{V^{1/2}JJ'V^{1/2}}{J'VJ}$$

is idempotent with $\text{tr}(C) = p - 1$ and $J'V^{1/2}C = 0$, $Q = Z'CZ$ has a chi-square distribution with $p - 1$ degrees of freedom and non-centrality parameter $\Delta^2_F = n(\theta' V^{-1}\theta - \theta'JJ'\theta/J'VJ)$ and Q and L are independent, see Rao (1973, pp. 186, 209). Conditioning on L, the power of χ^2_+ with critical value $c > 0$ is

$$\begin{aligned}\pi_F &= P[Q + L^2 > c \text{ and } L > 0] \\ &= \int_0^{\sqrt{c}} \phi\left(u - \frac{\sqrt{n}J'\theta}{\sqrt{J'VJ}}\right) H(c - u^2; p-1, \Delta^2_F)\, du + \Phi\left(\frac{\sqrt{n}J'\theta}{\sqrt{J'VJ}} - \sqrt{c}\right)\end{aligned} \tag{2.9}$$

where $H(\cdot, p - 1, \Delta^2_F)$ is the survival function of a chi-square distribution with $p - 1$ degrees of freedom and non-centrality parameter Δ^2_F. In our power study, (2.9) was computed by numerical integration.

2.4 A modification of Follmann's test

Because the second term in (2.8) does not involve the covariance matrix, V, we consider a modification of Follmann's test. With Z and η defined as in the last two subsections, the new test, which is denoted Z^2_+, rejects H_0 at level α if

$$Z'Z > \chi^2_{2\alpha,p} \text{ and } \sum_{j=1}^{p} Z_j > 0.$$

This is Follmann's test of $\eta_1 = \eta_2 = \cdots = \eta_p = 0$ based on Z, and thus its level of significance is α. Its power function is given by (2.9) with θ and V replaced by η and I, and Δ^2_F becomes $\Delta^2_N = \sum_{j=1}^{p} (\eta_j - \bar{\eta})^2$ with $\bar{\eta} = \sum_{j=1}^{p} \eta_j/p$.

For some θ and V, $J'\eta = \sqrt{n}J'V^{-1/2}\theta$ may be smaller than $\sqrt{n}J'\theta$, and one would prefer χ^2_+ over Z^2_+ in such situations. As expected, in all the cases considered in the power study, $\pi_F > \pi_N$ if $P[J'\bar{X} > 0] > P[J'Z > 0]$ and vice versa. For a given V, if one wants to choose the one of these two tests which performs better over all of H_1, one could select χ^2_+ if the minimum of $P[J'\bar{X} > 0]$ over $\theta = e_i$, where e_i are the p unit vectors, is larger than the minimum of $P[J'Z > 0]$ over $\theta = e_i$, and select Z^2_+ otherwise.

For the special $p \times p$ covariance matrix V, which has diagonal elements equal to one and all off-diagonal element equal to ρ, we show that χ^2_+ and Z^2_+ are identical. Johnson and Wichern (1992, pp. 365–367) showed that V has eigenvalues $\lambda_i = 1 - \rho$, $i = 1, 2, \ldots, p - 1$, and $\lambda_p = 1 + (p - 1)\rho$. The corresponding orthogonal, normalized eigenvectors are $e_i = (e_{i1}, e_{i2}, \ldots, e_{ip})'$ where for $1 \leq i \leq p - 1$, $e_{ij} = -1/\sqrt{i^2 + i}$ for $1 \leq j \leq i$, $e_{i,i+1} = i/\sqrt{i^2 + i}$, $e_{ij} = 0$ for $j > i + 1$ and $e_{pj} = 1/\sqrt{p}$ for $1 \leq j \leq p$. Also, see for example Johnson and Wichern (1992, p. 53),

$$V^{-1/2} = \sum_{i=1}^{p-1} \frac{1}{\sqrt{1-\rho}} e_i e_i' + \frac{1}{\sqrt{1+(p-1)\rho}} e_p e_p'.$$

However, the sum of elements in each column (and each row) of $e_i e_i'$ is zero for $1 \leq i \leq p-1$ and the sum of elements in each column (and each row) of $e_p e_p'$ is 1. Thus, the summation of each column of $V^{1/2}$ is the same, $d(\bar{X}_1 + \bar{X}_2 + \cdots + \bar{X}_p) = Z_1 + Z_2 + \cdots + Z_p$ where d is a positive constant, and χ^2_+ and Z^2_+ are identical.

For any p with $\rho = 0.5$, this V is the covariance matrix for the simple tree order with equal weights, see (1.2), and for $p = 2$ with $\rho = -0.5$, it is the covariance matrix for the simple order with equal weights, see (1.1).

We also considered the version of Z^2_+ obtained by replacing $V^{-1/2}$ by the A developed by Tang et al. (1989). In our power study, there were only slight differences in the powers of these two tests except in the cases in which V has a mixture of positive and negative correlations. In these exceptional cases, neither test is clearly superior to the other. However, as is pointed out in Section 2.6, we do not recommend Z^2_+ (either version) in these cases. Because $V^{-1/2}$ is simpler to compute than A, we report powers for the version of Z^2_+ based on $V^{-1/2}$.

2.5 A permutation test

For data arising from matched pairs and independent random samples, Boyett and Shuster (1977) proposed permutation procedures as non-parametric tests of H_0 versus $H_1 - H_0$. For matched pairs with the differences assumed to be normally distributed, we compared the power of the Boyett–Shuster test with Kudo's test for the simple order and the simple tree order by Monte-Carlo techniques. For the case of known variances, we replaced the denominator of their t_j by the standard deviation of the jth coordinate of X_i. For the permutation test to be effective n must be greater than one. We considered $n = 6$, 20 and 100. For $n = 20$ and 100 we did not consider all 2^n elements in their E_1. Only 1000 randomly selected elements of E_1 were used to approximate the significance level.

With $p = 3$, we chose mean vectors in the non-negative orthant so that the usual χ^2 test has power of 0.70 with a level of significance of 0.05. For the simple order, with $\theta = 0.3828(1, 1, 1)$ and $n = 6$; $\theta = 0.2097(1, 1, 1)$ and $n = 20$; and $\theta = 0.0938(1, 1, 1)$ and $n = 100$, Boyett and Shuster's test has estimated powers 0.289, 0.338 and 0.342 which are smaller than the other tests considered here. In fact, Kudo's test has powers 0.879, 0.873 and 0.878, respectively. For the simple tree order, with $\theta = 0.9884(1, 0, 0)$ and $n = 6$; $\theta = 0.5414(1, 0, 0)$ and $n = 20$; and $\theta = 0.2421(1, 0, 0)$ and $n = 100$, Boyett and Shuster's test has the smallest powers, i.e. 0.540, 0.617 and 0.636 while Kudo's test has powers 0.755, 0.756 and 0.753, respectively. The directions presented here are those for which the Boyett–Shuster test performed poorest.

We do not consider the Boyett–Shuster test further. However, it should be considered if the assumption of normality were in question.

2.6 Power comparisons

To compare the performance of $\bar{\chi}^2_{01}$, χ^2_+, Z^2_+ and T, we considered the simple order, and the simple tree order with equal weights, i.e. $n_1/\sigma_1^2 = n_2/\sigma_2^2 = \cdots = n_k/\sigma_k^2$, for $k = 3$, 4 and 7 ($p = k-1$), as well as some other forms of covariance structures. The correlation matrices of differences for the simple order and the simple tree are given in (1.1) and (1.2), respectively. We consider mean vectors of the form, $\theta = dv$ with d a constant and v a vector, refer to the vector v as a direction, and choose d so that the usual χ^2 test has power = 0.70 provided $v \neq 0$. For example, for $p = 2$ we consider directions (0, 0), (0, 1), (1, 0) and (1, 1). Since without loss of generality, one can take $n = 1$, we generated 10,000 multivariate normal X's and recorded the proportion of rejections for $\bar{\chi}^2_{01}$, χ^2_+, Z^2_+ and T. All these tests are conducted using the level of significance $\alpha = 0.05$. We also computed the exact powers of these tests for $p = 2$ and 3. Let π_K be the exact power of $\bar{\chi}^2_{01}$, $\hat{\pi}_K$ be the proportion of rejections for χ^2_{01} for the Monte-Carlo method, and let π_F, $\hat{\pi}_F$, π_N, $\hat{\pi}_N$, π_T and $\hat{\pi}_T$ be defined similarly.

The results of the power computations are given in Tables 1–3. With $p = 3$, we also considered V with $V_{ii} = 1$ for $1 \leq i \leq 3$, $V_{12} = V_{23} = -0.4$ and $V_{13} = 0.4$. The results are similar to those in Table 2 with $V_{12} = V_{23} = V_{13} = -0.4$. In particular, Kudo's test is preferred, T is a close second and π_F and π_N are about the same. We now summarize the results of the power study.

- In every case for which expressions have been given for the power functions of those tests, i.e. $\bar{\chi}^2_{01}$ and T with $p = 2$ and 3 and χ^2_+ and Z^2_+ with $p = 2, 3$ and 6. The Monte-Carlo power estimates and the exact power values are in good agreement. The maximum difference is 0.008.

Table 2

Powers times 1000 of the tests for known V with $p = 3$ and $V_{ii} = 1$ for $1 \leq i \leq 3$

	$V_{12} = V_{23} = V_{13} = -0.4$				$V_{12} = V_{23} = 0.4$ and $V_{13} = -0.4$			
	Test				Test			
Direction	π_K	π_F	π_N	π_T	π_K	π_F	π_N	π_T
(1, 1, 1)	888	799	800	843	826	797	799	836
(1, 0, 1)	879	799	800	841	799	710	772	746
(0, 1, 1)	879	799	800	841	803	798	797	821
(1, 1, 0)	879	799	800	841	803	798	797	821
(1, 0, 0)	859	797	799	836	782	692	761	751
(0, 1, 0)	860	799	799	836	767	692	546	639
(0, 0, 1)	860	799	799	836	782	692	761	751

Table 3

Powers times 1000 of the tests for known V, including the simple and simple tree orders with equal weights and $p = 6$

	Simple order				Simple order			
	Test				Test			
Direction	$\hat{\pi}_K$	$\hat{\pi}_F$	$\hat{\pi}_N^*$	$\hat{\pi}_T$	$\hat{\pi}_K$	$\hat{\pi}_F$	$\hat{\pi}_N^*$	$\hat{\pi}_T$
(1, 1, 1, 1, 1, 1)	932	801	802	887	807	797	797	882
(0, 1, 1, 1, 1, 1)	927	799	803	887	794	800	800	780
(0, 0, 1, 1, 1, 1)	923	798	802	877	772	764	764	730
(0, 0, 0, 1, 1, 1)	912	799	803	872	755	710	710	706
(0, 0, 0, 0, 1, 1)	903	803	805	867	754	658	658	701
(0, 0, 0, 0, 0, 1)	884	804	801	851	739	578	578	712
	$V_{ij} = -0.1$ for $1 \leq i \neq j \leq 6$				$V_{13} = V_{15} = V_{16} = V_{23} = V_{24} = V_{34} = V_{56} = 0.4$ and $V_{ij} = -0.4$ for others with $i < j$			
	Test				Test			
Direction	$\hat{\pi}_K$	$\hat{\pi}_F$	$\hat{\pi}_N^*$	$\hat{\pi}_T$	$\hat{\pi}_K$	$\hat{\pi}_F$	$\hat{\pi}_N^*$	$\hat{\pi}_T$
(1, 1, 1, 1, 1, 1)	910	805	805	885	875	800	800	884
(0, 1, 1, 1, 1, 1)	903	804	804	881	844	729	749	703
(0, 0, 1, 1, 1, 1)	894	805	805	877	824	641	656	618
(0, 0, 0, 1, 1, 1)	878	801	801	864	833	759	779	758
(0, 0, 0, 0, 1, 1)	862	801	801	855	819	623	655	611
(0, 0, 0, 0, 0, 1)	836	785	785	835	795	604	638	634

Note: For each V, $V_{ii} = 1$ for $1 \leq i \leq 6$.

- For all covariance forms considered here, all four tests tend to have the highest power at the middle of the orthant with the power decreasing as the alternative moves away from the center of the orthant. That is, for $p = 3$ the powers tend to be highest for direction (1, 1, 1), smaller for (1, 0, 1), (0, 1, 1) and (1, 1, 0) and smallest at (1, 0, 0), (0, 1, 0) and (0, 0, 1).
- For $p \geq 3$ and V with several of the correlations positive, in particular for the simple tree; for $p = 3$, $V_{12} = V_{23} = 0.4$ and $V_{13} = -0.4$; and for $p = 6$, $V_{12} = V_{14} = V_{25} = V_{26} = V_{35} = V_{36} = V_{45} = V_{46} = -0.4$ and the other $V_{ij} = 0.4$ for $1 \leq i \neq j \leq 6$,

Follmann's and the new test have smaller powers for some directions than the usual χ^2 test. Thus, we do not recommend these tests in such cases. Also, in the latter two cases T has smaller powers than the usual χ^2 test and we do not recommend it in those cases.

- For each case considered, $\bar{\chi}^2_{01}$ has the best overall power among the four tests. In particular for each p and each covariance matrix, the minimum power over all the directions considered was largest for $\bar{\chi}^2_{01}$. Except for covariance matrices with several positive correlations, which includes the simple tree ordering, the power of $\bar{\chi}^2_{01}$ was essentially as large as or larger than that of the other three test for each direction considered. However, in these exceptional cases, T has the largest power at the center of the orthant, i.e. for $v = (1, 1,..., 1)$, but its power is substantially smaller than $\bar{\chi}^2_{01}$ at some of the edges of the orthant.
- Recall, if all correlations are equal then χ^2_+ and Z^2_+ have identical powers. Also, for all the cases considered here, except those with several positive and some negative correlations, χ^2_+ and Z^2_+ had similar powers, i.e. the largest difference is 0.006.
- Except for the cases in which there are several positive correlations (recall, we do not recommend χ^2_+ or Z^2_+ in these cases), T had powers as large as or larger than χ^2_+ and Z^2_+ for each direction considered. Also for the simple tree, we recommend T over χ^2_+ or Z^2_+ because, unlike χ^2_+ and Z^2_+, its power does not fall below those of the usual χ^2 test.
- For the simple order, the maximum loss in power when one uses χ^2_+ instead of $\bar{\chi}^2_{01}$ is 5.5% for p = 2, 9.0% for $p = 3$ and 14.0% for $p = 6$. The corresponding value for $p = 2$ and the simple tree is 2.9%, and for larger p we do not recommend the use of χ^2_+ for the simple tree.
- For the simple order, the maximum loss in power when one uses T instead of $\bar{\chi}^2_{01}$ is 2.6% for $p = 2$, 4.2% for $p = 3$, and 4.9% for $p = 6$. For the simple tree, the corresponding values are 3.4% for $p = 2$, 4.9% for $p = 3$, and for $p = 6$ the power of T is as small as the usual χ^2 test at some of the edges.

In summary, $\bar{\chi}^2_{01}$ has the best overall powers of the four tests. If V has few positive correlations, one could use T which has larger power than $\bar{\chi}^2_{01}$ at the center of the orthant, but substantially smaller powers at some of its edges. If V has several positive correlations, χ^2_+, Z^2_+ and T should not be used if θ may be near one of the edges of the orthant.

3. $V = \sigma^2 V_0$ with V_0 known

Let $X_i = (X_{i1}, X_{i2},..., X_{ip})' \sim N(\theta, V)$, $i = 1, 2,..., n$, where $V = \sigma^2 V_0$ with V_0 known and σ^2 unknown, let X_i be independent, and let H_0 and H_1 be defined as in Section 1. We extend Follmann's test and the Tang–Gnecco–Geller test to this setting and compare their powers with those of Shorack's test, the new test and Bartholomew's test. Because of the poor performance of the Boyett–Shuster test in the known covariance case, we did not consider it for partially known covariance matrices.

3.1 Shorack's test

In Robertson et al. (1988, p. 221), the LRT of H_0 versus $H_1 - H_0$ is shown to reject H_0 for large values of

$$\bar{E}^2 = \frac{n\theta^{*\prime} V_0^{-1}\theta^*}{\sum^n_{i=1} X_i' V_0^{-1} X_i}, \tag{3.1}$$

where θ^* is the restricted MLE of θ which is defined after (2.1). From their Theorem 4.6.2, under H_0 with t a real number

$$P(\bar{E}^2 \geqslant t) = \sum_{j=0}^{p} Q(j, p; V) P(B_{j/2,(np-j)/2} \geqslant t), \tag{3.2}$$

where $Q(j, p; V)$, $j = 0, 1, 2,..., p$, are the same level probabilities as in (2.2) and $B(a, b)$ denotes a random variable having a beta distribution with parameters a and $b(B(0, b) \equiv 0)$.

3.2 An approximate likelihood ratio test

We develop a version of the Tang–Gnecco–Geller test for $V = \sigma^2 V_0$. Let $Z_i = V_0^{-1/2} X_i \sim N(\eta, \sigma^2 I)$ with $\eta = V_0^{-1/2}\theta$.

From p. 221 of Robertson et al. (1988), the LRT of $\eta = 0$ versus $\eta_i \geqslant 0$, $i = 1, 2,..., p$, with at least one strict inequality rejects $\eta = 0$ for large values of

$$\bar{E}_a^2 = \frac{n\eta^{*\prime}\eta^*}{\sum_{i=1}^n Z_i' Z_i} = \frac{n\eta^{*\prime}\eta^*}{\sum_{i=1}^n X_i' V_0^{-1} X_i}. \tag{3.3}$$

Applying Theorem 4.6.2 of Robertson et al. (1988) and the null hypothesis distribution of Tang et al. (1989), under H_0 for any real number t,

$$P(\bar{E}_a^2 \geqslant t) = \sum_{j=0}^{p} \frac{p!}{j!(p-j)!} P(B_{j/2,(np-j)/2} \geqslant t)/2^p. \tag{3.4}$$

3.3 Follmann's test

We extend Follmann's test to the case in which the covariance matrix is known up to a multiplicative constant. One can show that the LRT of H_0 versus $\sim H_0$ rejects H_0 for large values of

$$F = \frac{n(n-1)\bar{X}'V_0^{-1}\bar{X}}{\sum_{i=1}^n (X_i - \bar{X})'V_0^{-1}(X_i - \bar{X})}. \tag{3.5}$$

Of course, $\sum_{i=1}^n (X_i - \bar{X})'V_0^{-1}(X_i - \bar{X})/\sigma^2$ and $\bar{X}$ are independent, the first has a chi-squared distribution with degrees of freedom $p(n-1)$ and $n(\bar{X} - \theta)'V_0^{-1}(\bar{X} - \theta)/\sigma^2$ has a chi-squared distribution with p degrees of freedom. Hence, under H_0, $F \sim F_{p,(n-1)p}$, i.e. the F-distribution with degrees of freedom p and $(n-1)p$. Therefore, for $V = \sigma^2 V_0$, Follmann's test, which we denote by F_+, rejects H_0 if

$$F > F_{2\alpha;p,(n-1)p} \text{ and } \sum_{j=1}^{p} \bar{X}_j > 0, \tag{3.6}$$

where $F_{2\alpha;p,(n-1)p}$ is the $(1-2\alpha)$th quantile of the F-distribution with degrees of freedom p and $(n-1)p$. The significance level of Follmann's test for this case is considered next. Under H_0, let $T = \sum_{i=1}^n (X_i - \bar{X})'V_0^{-1}(X_i - \bar{X})/[\sigma^2(n-1)]$ and let $h(t)$ denote the density of T. By Theorem 2.1 of Follmann (1996), since $1'\theta = 0$, the significance level is

$$\int_0^\infty h(t) P\left[\frac{(n/\sigma^2)\bar{X}'V_0^{-1}\bar{X}}{t} > F_{2\alpha;p,(n-1)p} \mid T = t\right] P[J'\bar{X} > 0]\, dt = 2\alpha \times \tfrac{1}{2} = \alpha.$$

Next, an expression for the power function of F_+ is obtained. With critical value $c > 0$, conditioning on T and using the argument given in Section 2.3.

$$\pi_F(\theta) = \int_0^\infty h(t)\left[\int_0^{\sqrt{ct}} \phi\left(u - \frac{\sqrt{n}J'\theta}{\sqrt{J'V_0 J}}\right) \mathrm{H}(ct - u^2; p-1, \Delta_F^2) + \Phi\left(\frac{\sqrt{n}J'\theta}{\sqrt{J'V_0 J}} - \sqrt{ct}\right)\right] du\, dt, \tag{3.7}$$

where H and δ_F^2 are defined as in Section 2.3 with V replaced by V_0. For the cases considered in our power study, (3.7) was computed using QDAG in IMSL. These values and the Monte-Carlo estimates did not differ by more than 0.008.

3.4 The new test

If $Z_i = V_0^{-1/2} X_i$, and $\bar{Z} = (Z_1 + Z_2 + \cdots + Z_n)/n$ then $\bar{Z} = V_0^{-1/2}\bar{X}$, $n\bar{X}' V_0^{-1}\bar{X} = n\bar{Z}'\bar{Z}$ and $(X_i - \bar{X})'V_0^{-1}(X_i - \bar{X}) = (Z_i - \bar{Z})'(Z_i - \bar{Z})$. The modified Follmann test, denoted G_+, rejects H_0 if

$$G = \frac{n(n-1)\bar{Z}'\bar{Z}}{\sum_{i=1}^n (Z_i - \bar{Z})'(Z_i - \bar{Z})} > F_{2\alpha;p,(n-1)p} \text{ and } \sum_{j=1}^{p} \bar{Z}_j > 0. \tag{3.8}$$

Because $Z_i \sim N(V_0^{-1/2}\theta, \sigma_i^2)$, this test is Follmann's test of $\eta = V_0^{-1/2}\theta = 0$ versus $\eta \geqslant 0$ with at least one strict inequality and has significance level α. If all of the diagonal elements of V_0 are equal and all of the off-diagonal elements of V_0 are equal, then as in Section 2, F_+ and G_+ are identical. Also, the power functions of G_+ can be obtained from (3.7) by replacing θ and V_0 by $V_0^{-1/2}\theta$ and I, respectively.

3.5 Bartholomew's test

Bartholomew's tests of homogeneity of normal means with an order restricted alternative are discussed in Chapter 2 of Robertson et al. (1988). It was noted in the

Table 4

With $V = \sigma^2 V_0$, powers times 1000 of the tests for the simple order and simple tree with equal weights and $p = 3$

Simple order, $p = 3$						Simple tree, $p = 3$				
Test						Test				
Direction	$\hat{\pi}_S$	$\hat{\pi}_F$	$\hat{\pi}_N$	$\hat{\pi}_B$	$\hat{\pi}_T$	$\hat{\pi}_S$	$\hat{\pi}_F$	$\hat{\pi}_N$	$\hat{\pi}_B$	$\hat{\pi}_T$
	$n = 6$					$n = 6$				
(1, 1, 1)	916	820	820	921	876	829	814	814	846	877
(1, 0, 1)	904	818	818	913	877	807	807	807	817	756
(0, 1, 1)	905	823	823	909	872	804	808	808	819	758
(1, 1, 0)	903	819	819	911	869	802	803	803	824	752
(1, 0, 0)	874	813	812	887	860	778	722	722	805	730
(0, 1, 0)	887	820	823	892	872	782	721	721	798	732
(0, 0, 1)	880	817	817	886	866	778	719	719	796	728
	$n = 20$					$n = 20$				
(1, 1, 1)	885	803	803	896	847	809	802	802	815	852
(1, 0, 1)	874	802	802	885	848	781	778	778	784	748
(0, 1, 1)	874	802	803	881	841	778	776	776	792	748
(1, 1, 0)	876	801	802	885	846	780	781	781	787	750
(1, 0, 0)	852	800	800	858	836	764	697	697	768	728
(0, 1, 0)	856	792	798	864	839	761	696	696	769	724
(0, 0, 1)	846	801	800	859	834	760	693	693	767	722

Introduction that if the variances are known, then Bartholomew's and Kudo's tests are equivalent for the simple order and the simple tree order. In Section 3.6, we compare Bartholomew's and Shorack's tests when the variances are known up to a multiplicative constant.

3.6 Power comparisons

The performance of these five tests are studied by Monte-Carlo techniques for the correlation matrices (1.1) and (1.2), that is for the simple order and the simple tree order with equal weights, i.e. $n_1/\sigma_1^2 = n_2/\sigma_2^2 = \cdots = n_k/\sigma_k^2$, and $k = 4$ and 7 ($p = k - 1$) as well as some other covariance matrices. As before, the mean vector was chosen in the non-negative orthant so that the usual F test has power = 0.70. We used 10,000 iterations. In each iteration, n multivariate normal X's with the chosen mean vector and covariance of the form $\sigma^2 V_0$ were generated and the proportion of rejections for these tests were recorded. Since all of the tests are scale invariant, we chose $\sigma^2 = 1$. All of these tests are conducted using the level of significance $\alpha = 0.05$. Let $\hat{\pi}_S$, $\hat{\pi}_F$, $\hat{\pi}_N$, $\hat{\pi}_B$ and $\hat{\pi}_T$ denote the proportion of rejections for Shorack's test, Follmann's test, the new test, Bartholomew's test and the Tang–Gnecco–Geller test. We considered a small n ($n = 6$ for $p = 3$ and $n = 10$ for $p = 6$), $n = 20$ and 100. Because the results for $n = 100$ are like those in Section 2, we did not include these power estimates in the tables.

Monte-Carlo estimates of power are given in Tables 4–7. In addition to the results presented in those tables, we also considered the covariance matrix V with $p = 3$, $V_{ii} = 1$ for $1 \leqslant i \leqslant p$, $V_{12} = V_{23} = -0.4$ and $V_{13} = 0.4$. However, the

Table 5

With $V = \sigma^2 V_0$, powers times 1000 of the tests when $V_0 = [V_{ij}]$, $V_{ii} = 1$ for $1 \leqslant i \leqslant p$ and $p = 3$

$V_{ij} = -0.4$ for $1 \leqslant i \neq j \leqslant 3$					$V_{12} = V_{23} = 0.4$ and $V_{13} = -0.4$			
Test					Test			
Direction	$\hat{\pi}_S$	$\hat{\pi}_F$	$\hat{\pi}_N$	$\hat{\pi}_T$	$\hat{\pi}_S$	$\hat{\pi}_F$	$\hat{\pi}_N$	$\hat{\pi}_T$
	$n = 6$				$n = 6$			
(1, 1, 1)	926	821	821	878	858	816	817	868
(1, 0, 1)	917	821	821	878	833	752	806	750
(0, 1, 1)	919	821	821	876	834	821	821	856
(1, 1, 0)	917	820	820	874	833	817	816	850
(1, 0, 0)	898	814	814	869	811	736	798	757
(0, 1, 0)	900	823	823	874	799	734	577	607
(0, 0, 1)	899	818	818	874	813	732	793	765
	$n = 20$				$n = 20$			
(1, 1, 1)	898	801	801	849	831	798	799	842
(1, 0, 1)	889	801	801	846	800	717	774	746
(0, 1, 1)	887	799	799	845	810	800	799	828
(1, 1, 0)	887	801	801	847	806	801	800	829
(1, 0, 0)	868	799	799	840	785	703	767	750
(0, 1, 0)	868	799	799	843	778	710	557	636
(0, 0, 1)	867	802	802	843	786	702	767	751

conclusions are like those for V with $p = 3$, $V_{ii} = 1$ for $1 \leqslant i \leqslant p$ and $V_{ij} = -0.4$ for $1 \leqslant i \neq j \leqslant p$. In particular, Shorack's test is preferred, the Tang–Gnecco–Geller test is a close second and Follmann's test and the new test have about the same powers. The results from the power study are summarized below.

- For all five tests and all the covariance matrices considered, the Monte-Carlo power estimates at the null hypothesis are close to 0.05. The maximum difference is 0.004.
- For all covariance forms considered here, all five tests tend to have the highest power at the middle of the orthant with the power decreasing as the alternative moves away from the center of the orthant.
- For a given p and covariance matrix V, the powers tend to be larger for smaller n. There is one exception, see the right half of Table 5 with direction (0, 1, 0), which may be due to Monte-Carlo error. This says that, as would be expected, the gain in power due to using the restricted alternative is greater for smaller n.
- For $p = 3$, $p = 6$ and small n, with the simple and simple tree orders, Bartholomew's test has the largest and Shorack's test has the second largest power for each direction considered, except that for the simple tree order, the Tang–Gnecco–Geller test has larger powers at the center but its powers are substantially smaller than Bartholomew's test for some other directions. The loss in power due to using Shorack's test in place of Bartholomew's test is greater for the simple tree than for the simple order and is greater for small p and small n. For instance, for the simple tree, $p = 3$ and $n = 6$, the loss is about 3.4%, and for the simple order, $p = 3$ and $n = 6$, it is about 1.5%. For both orders with $p = 6$ and $n \geqslant 20$, there is little difference in the powers of these two tests.

- As in Section 2, for covariance matrices with several positive correlations, Follmann's test and the new test have, for some directions, powers as small as or smaller than the usual F test for unordered alternatives, except that for $p = 3$ and $n = 6$, Follmann's test does not have this shortcoming. However, in these cases, the power of Follmann's test is as much as 10.0% below that of Shorack's test. Thus, we do not recommend Follmann's test or the new test for covariance matrices

Table 6

With $V = \sigma^2 V_0$, powers times 1000 of the tests for the simple order and simple tree with equal weights and $p = 6$

Simple order Test						Simple tree Test				
Direction	$\hat{\pi}_S$	$\hat{\pi}_F$	$\hat{\pi}_N$	$\hat{\pi}_B$	$\hat{\pi}_T$	$\hat{\pi}_S$	$\hat{\pi}_F$	$\hat{\pi}_N$	$\hat{\pi}_B$	$\hat{\pi}_T$
	$n = 10$					$n = 10$				
(1, 1, 1, 1, 1, 1)	945	803	804	949	896	817	802	802	828	896
(0, 1, 1, 1, 1, 1)	940	803	805	944	891	791	797	797	806	774
(0, 0, 1, 1, 1, 1)	934	799	802	939	888	773	771	771	791	718
(0, 0, 0, 1, 1, 1)	925	802	805	930	884	761	727	727	779	696
(0, 0, 0, 0, 1, 1)	914	801	803	919	874	753	670	670	766	695
(0, 0, 0, 0, 0, 1)	896	800	798	899	863	743	588	588	750	710
	$n = 20$					$n = 20$				
(1, 1, 1, 1, 1, 1)	935	800	801	936	889	812	800	800	819	893
(0, 1, 1, 1, 1, 1)	932	796	799	933	885	791	794	794	793	774
(0, 0, 1, 1, 1, 1)	928	798	802	927	881	770	765	765	776	723
(0, 0, 0, 1, 1, 1)	922	802	805	918	878	758	718	718	760	697
(0, 0, 0, 0, 1, 1)	911	804	806	906	872	750	657	657	748	692
(0, 0, 0, 0, 0, 1)	889	801	798	885	861	743	580	580	741	709

Table 7

With $V = \sigma^2 V_0$, powers times 1000 of the tests when $V_0 = [V_{ij}]$, $V_{ii} = 1$ for $1 \leq i \leq p$ and $p = 6$

$V_{ij} = -0.1$ for $1 \leq i \neq j \leq 6$ Test					$V_{13} = V_{15} = V_{16} = V_{23} = V_{24} = V_{34} = V_{56} = 0.4$ and $V_{ij} = -0.4$ for others with $i < j$ Test			
Direction	$\hat{\pi}_S$	$\hat{\pi}_F$	$\hat{\pi}_N$	$\hat{\pi}_T$	$\hat{\pi}_S$	$\hat{\pi}_F$	$\hat{\pi}_N$	$\hat{\pi}_T$
	$n = 10$				$n = 10$			
(1, 1, 1, 1, 1, 1)	936	803	803	895	891	805	805	902
(0, 1, 1, 1, 1, 1)	916	803	803	894	849	738	754	687
(0, 0, 1, 1, 1, 1)	903	802	802	883	826	645	664	593
(0, 0, 0, 1, 1, 1)	890	805	805	878	846	765	785	762
(0, 0, 0, 0, 1, 1)	872	805	805	865	820	630	665	584
(0, 0, 0, 0, 0, 1)	846	785	785	845	804	611	647	614
	$n = 20$				$n = 20$			
(1, 1, 1, 1, 1, 1)	914	801	801	889	882	803	803	893
(0, 1, 1, 1, 1, 1)	904	799	799	883	843	727	747	700
(0, 0, 1, 1, 1, 1)	898	802	802	878	821	641	658	612
(0, 0, 0, 1, 1, 1)	881	802	802	869	838	762	784	769
(0, 0, 0, 0, 1, 1)	864	802	802	858	815	628	663	607
(0, 0, 0, 0, 0, 1)	839	783	783	839	799	609	643	637

with several positive correlations. The Tang–Gnecco–Geller test has the same difficulty as Follmann's test for the simple tree with $p = 6$. Thus, we do not recommend it for the simple tree with moderate p.

- For all the covariance matrices considered, besides those for the simple order and simple tree, Shorack's test has the best overall powers, that is its minimum power over all the directions considered is larger than that of the other tests. For covariance matrices with several positive correlations, the Tang–Gnecco–Geller test has the largest powers at the center of the orthant, but substantially smaller powers than Shorack's test on the edges of the orthant.

- Recall, if all the diagonal elements of V are the same and all of the off-diagonal elements of V are the same, then Follmann's test and the new test have identical powers. In all of the cases considered here, except for some with several positive correlations, the power estimates for Follmann's test and the new test are close, i.e. they do not differ by more than 0.006. Recall, we do not recommend Follmann's test or the new test when there are several positive correlations in V.
- For all the cases considered here, except the V with several positive correlations, the Tang–Gnecco–Geller test has powers as large as or larger than Follmann's test and the new test for each direction considered. Recall, we do not recommend Follmann's test or the new test when there are several positive correlations.
- For the simple order, the maximum loss in power when one uses Follmann's test instead of Bartholomew's test is 11.0% for $p = 3$ and 15.4% for $p = 6$. However, for the simple tree we do not recommend the use of Follmann's test.
- For the simple order, the maximum loss in power when one uses the Tang–Gnecco– Geller test instead of Bartholomew's test is 5.5% for $p = 3$ and 5.6% for $p = 6$. The corresponding value for the simple tree and $p = 3$ is 9.3%, and for the simple tree with $p = 6$, we do not recommend the use of the Tang–Gnecco–Geller test.

In summary, Bartholomew's test should be used for the simple and simple tree orders. However, there is little difference in the powers of Bartholomew's and Shorack's test for $n \geqslant 20$. For other covariance matrices, Shorack's test has the best overall power. If V has few positive correlations, one could use the Tang–Gnecco–Geller test which has larger power than Shorack's test at the center of the orthant, but substantially smaller powers at the edges. If V has several positive correlations, Follmann's test, the new test and the Tang–Gnecco–Geller test should not be used if the mean is near one of the edges of the orthant.

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References

Bartholomew, D.J., 1959a. A test of homogeneity for ordered alternatives. Biometrika 46, 46–48.

_______. 1959b. A test of homogeneity for ordered alternatives II. Biometrika 46, 328–335.

_______. 1961. A test of homogeneity of means under restricted alternatives (with discussion). J. Roy. Statist. Soc. (B) 23, 239–281.

Bohrer, R., Chow, W., 1978. Weights for one-sided multivariate inferences. Appl. Statist. 24, 380–384.

Boyett, J.M., Shuster, J.J., 1977. Nonparametric one-sided tests in multivariate analysis with medical applications. J. Amer. Statist. Assoc. 72, 665–668.

Chongcharoen, S., 1998. One-sided multivariate tests. Ph.D. Dissertation, University of Missouri-Columbia.

Conaway, M., Pillers, C., Robertson, T., Sconing, J., 1991. A circular cone test for testing homogeneity against a simple tree order. Can. J. Statist. 19, 283–296.

Follmann, D., 1996. A simple multivariate test for one-sided alternatives. J. Amer. Statist. Assoc. 91, 854–861.

Johnson, R.A., Wichern, D.W., 1992. Applied Multivariate Statistical Analysis, 3rd Edition. Simon & Schuster, New York.

Kudo, A., 1963. A multivariate analogue of the one-sided test. Biometrika 50, 403–418.

Rao, C.R., 1973. Linear Statistical Inferences and its Applications, 2nd Edition. Wiley, New York.

Robertson, T., Wright, F.T., Dykstra, R.L., 1988. Order Restricted Statistical Inference. Wiley, New York.

Shorack, G.R., 1967. Testing against ordered alternatives in model I analysis of variance: normal theory and nonparametrics. Ann. Math. Statist. 38, 1740–1753.

Sun, H.J., 1988. A fortran subroutine for computing normal orthant probability. Comm. Statist. Simulation Comput. 17, 1097–1111.

Tang, D., Lin, S.P., 1997. An approximate likelihood ratio test for comparing several treatments to a control. J. Amer. Statist. Assoc. 92, 1155–1162.

Tang, D., Gnecco, C., Geller, N.L., 1989. An approximate likelihood ratio test for a normal mean vector with nonnegative components with application to clinical trials. Biometrika 76, 577–583.